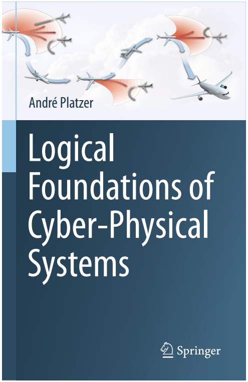
15: Winning Strategies & Regions

Logical Foundations of Cyber-Physical Systems



André Platzer

Carnegie Mellon University
Computer Science Department

Outline

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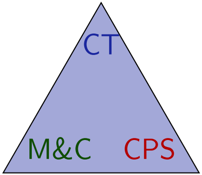
1. Learning Objectives
2. Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics
3. Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly
4. Summary

Learning Objectives

Winning Strategies & Regions

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
denotational vs. operational semantics



adversarial dynamics
adversarial semantics
adversarial repetitions
fixpoints

CPS semantics
multi-agent operational-effects
mutual reactions
complementary hybrid systems

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

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All Reals | Some Reals

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"Angel has Wings $\langle\alpha\rangle$"

All Reals | Some Reals | Angel Wins | Demon Wins

Outline

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[\![\neg P]\!] &= ([\![P]\!])^{\complement} \\
[\![P \land Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
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[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Only for HPs. No interactive play!

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$\varsigma_{x:=e}(X) =$

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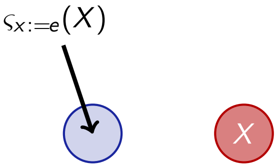
$$\varsigma_{x:=e}(X) = \{\omega \in \mathcal{S} : \omega_x^{\omega[\![e]\!]} \in X\}$$



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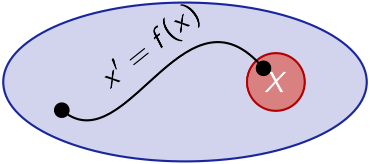

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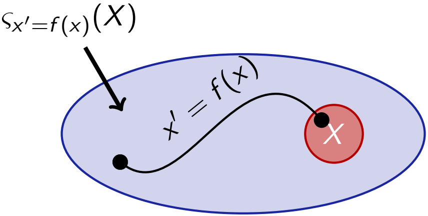

Differential Game Logic: Denotational Semantics

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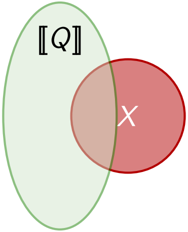

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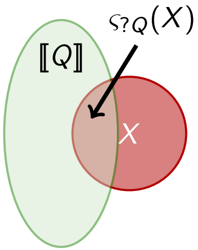

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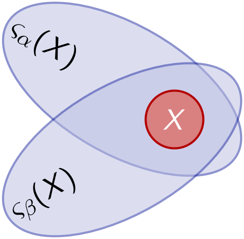

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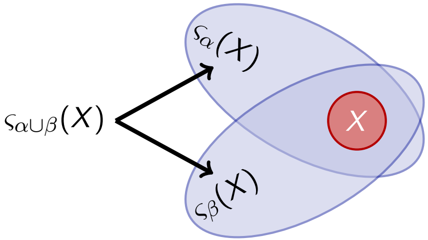

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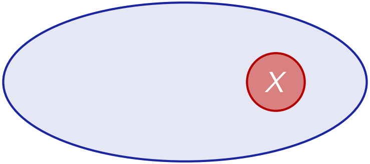

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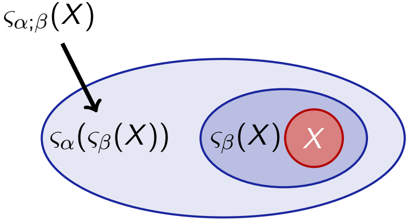

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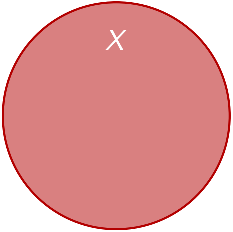

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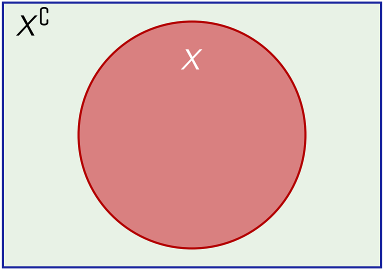

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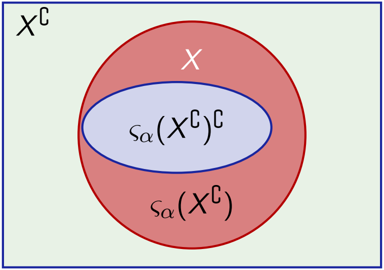

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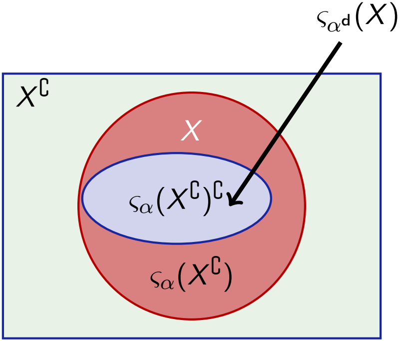

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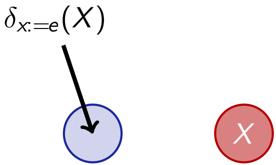

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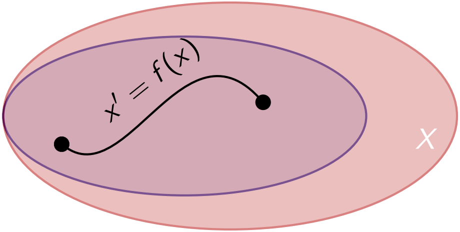

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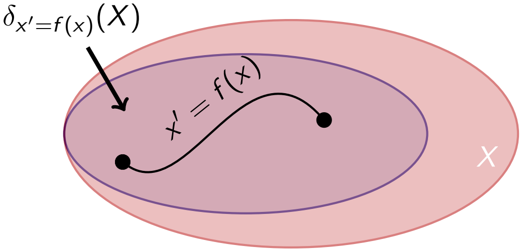

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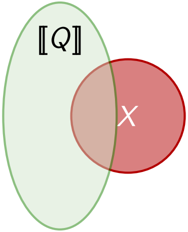

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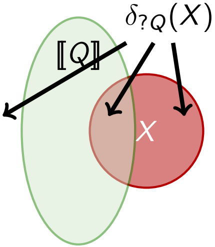
$$\delta_{?Q}(X) = [\![Q]\!]^\complement \cup X$$



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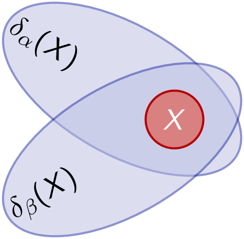

Differential Game Logic: Denotational Semantics

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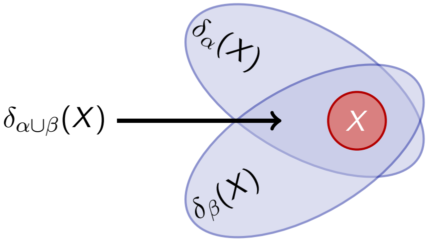

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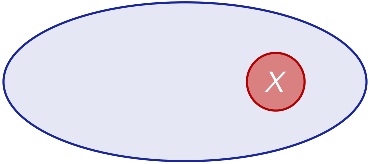

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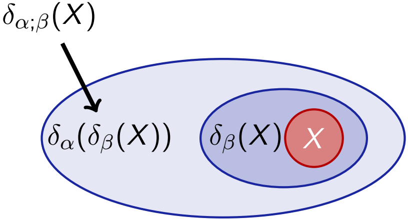

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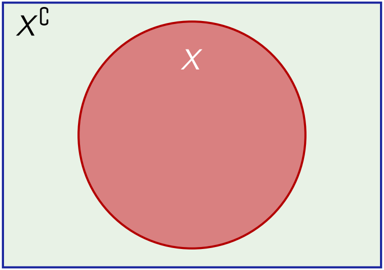

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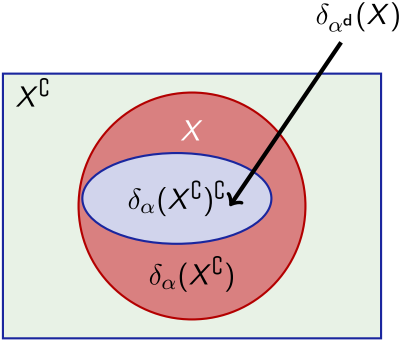

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\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
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[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
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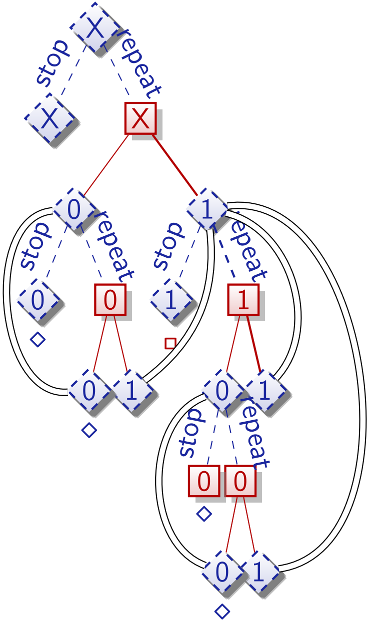
Outline

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



Semantics of Repetition

Definition (Hybrid game α)

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$$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$$

$$[\![\alpha^*]\!] = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!] \quad \text{where } \alpha^{n+1} \equiv \alpha^n; \alpha \quad \alpha^0 \equiv ?\mathit{true} \quad \text{for HP } \alpha$$

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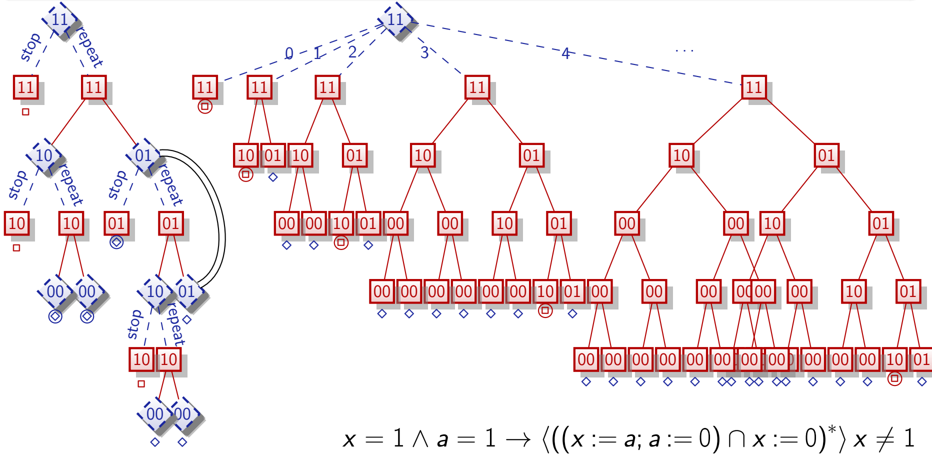
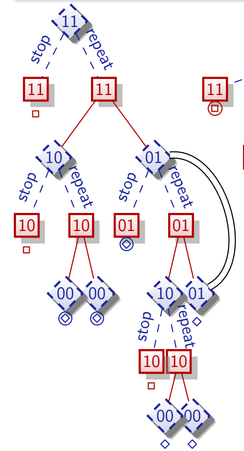
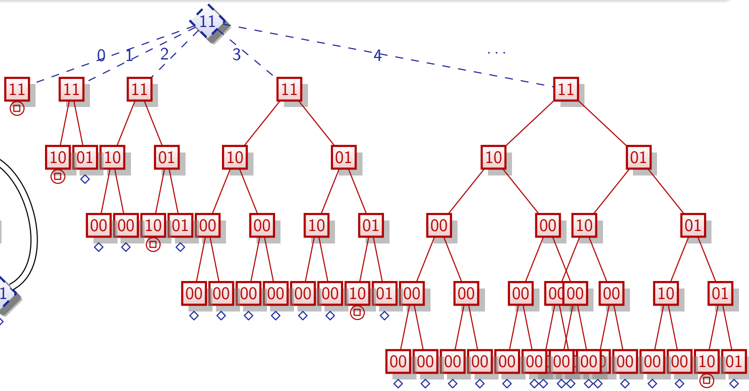


$$x = 1 \land a = 1 \rightarrow \langle((x := a; a := 0) \cap x := 0)^*\rangle\, x \neq 1$$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ advance notice semantics?





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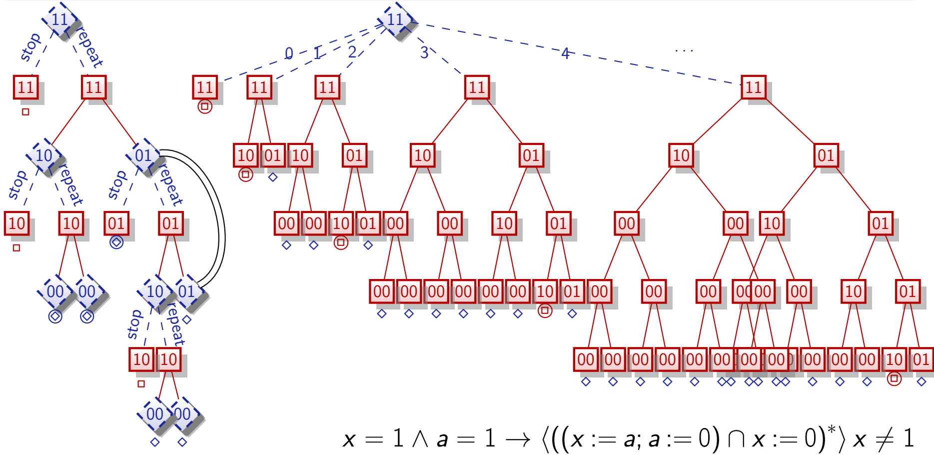
Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_{\alpha^n}(X)$ too hard to predict all iterations!



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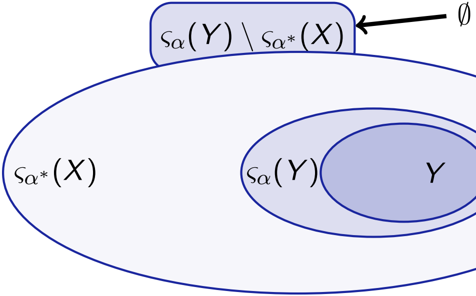
Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_{\alpha}(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_{\alpha}(Y)$ is just one round away from Y.



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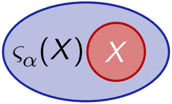

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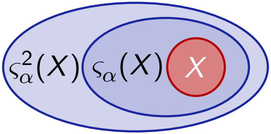

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n(X)$ *n* outside the game so Demon won't know

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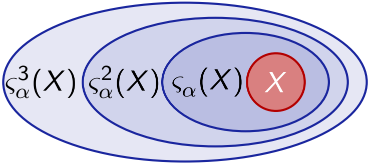

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Example

$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle \, (0 \le x < 1)$

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n(X)$

$$\varsigma_\alpha^0(X) \stackrel{\text{def}}{=} X$$

$$\varsigma_\alpha^{\kappa+1}(X) \stackrel{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$$

Example

$\langle(x := 1; x' = 1^{\mathrm{d}} \cup x := x - 1)^*\rangle\,(0 \le x < 1)$ $\qquad \varsigma_\alpha^n([0,1)) = [0,n) \neq \mathbb{R}$

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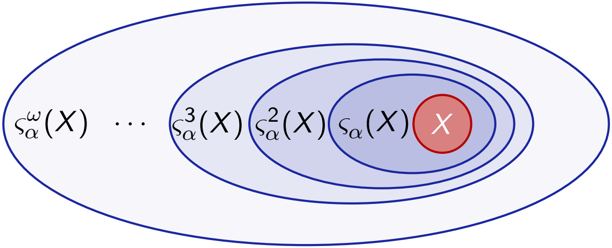

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{n\in\mathbb{N}} \varsigma_\alpha^n(X)$ missing winning strategies

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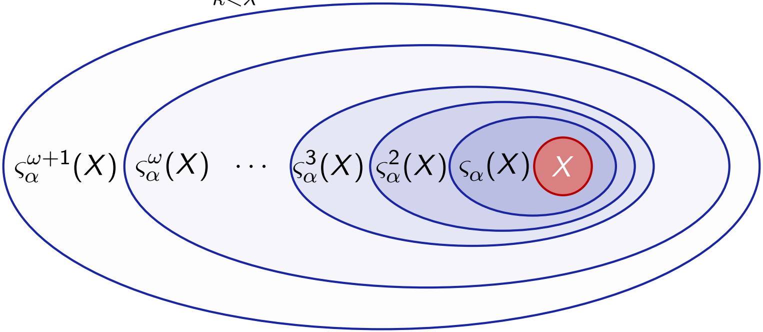
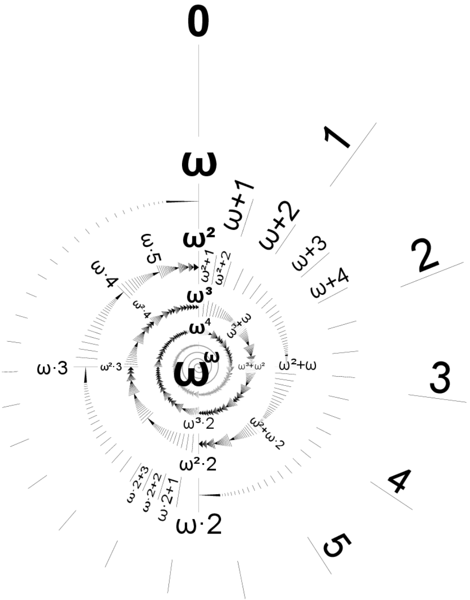


Theorem

Hybrid game closure ordinal $>\omega^\omega$



$$\iota + 0 = \iota$$

$$\iota + (\kappa+1) = (\iota + \kappa) + 1 \qquad \text{successor } \kappa+1$$

$$\iota + \lambda = \bigsqcup_{\kappa<\lambda} \iota + \kappa \qquad \text{limit } \lambda$$

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$$\iota \cdot (\kappa+1) = (\iota \cdot \kappa) + \iota \qquad \text{successor } \kappa+1$$

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$$\iota^{\kappa+1} = \iota^\kappa \cdot \iota \qquad \text{successor } \kappa+1$$

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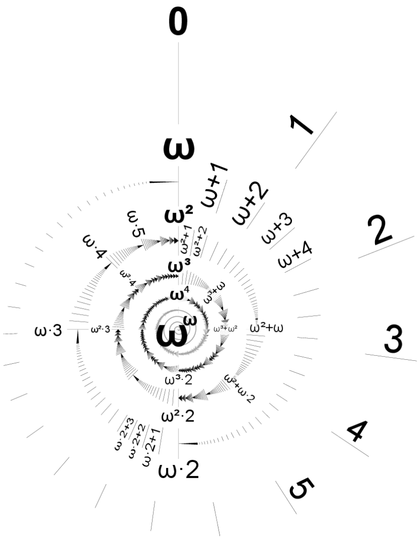
$$2 \cdot \omega = 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4$$



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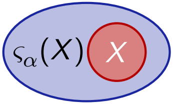

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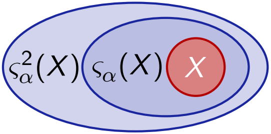

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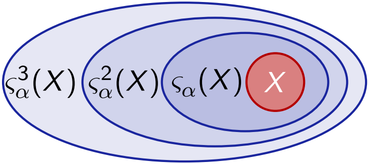

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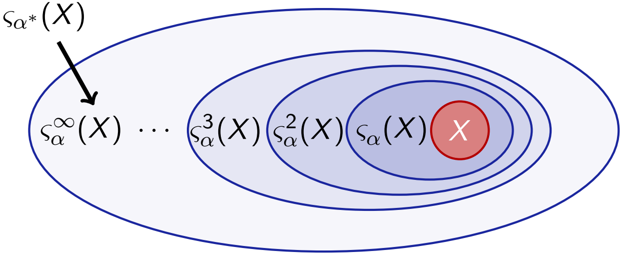
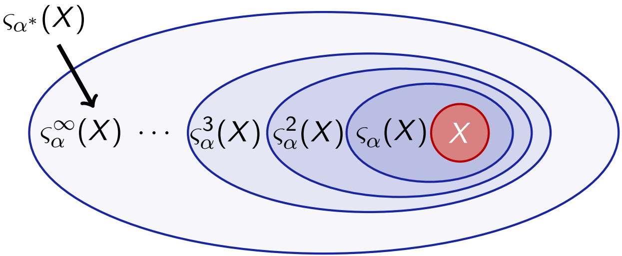


Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ requires transfinite patience



The Power of Implicit Definitions

Implicit Definitions

The advantages of implicit definition over construction are roughly those of theft over honest toil.

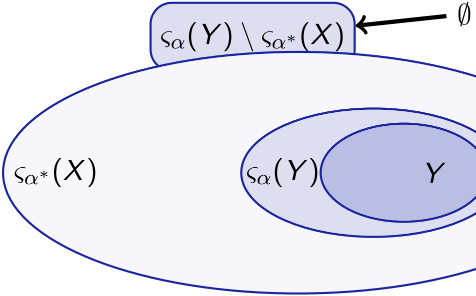
— Bertrand Russell

+1 Argument

Note (+1 argument)

$$Y \subseteq \varsigma_{\alpha^*}(X) \text{ then } \varsigma_\alpha(Y) \subseteq \varsigma_{\alpha^*}(X)$$

Since $\varsigma_\alpha(Y)$ is just one round away from Y.



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- Are there multiple such Z?
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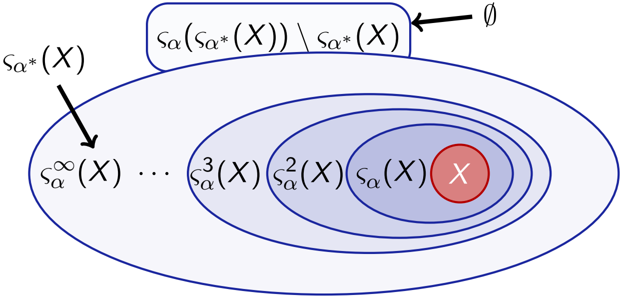
- Still too small: $X \subseteq Z$ since Angel may decide not to repeat

Fixpoints and Pre-Fixpoints

Definition (Pre-fixpoint)

$X \cup \varsigma_\alpha(Z) \subseteq Z$ for the winning region $Z \overset{\text{def}}{=} \varsigma_{\alpha^*}(X)$



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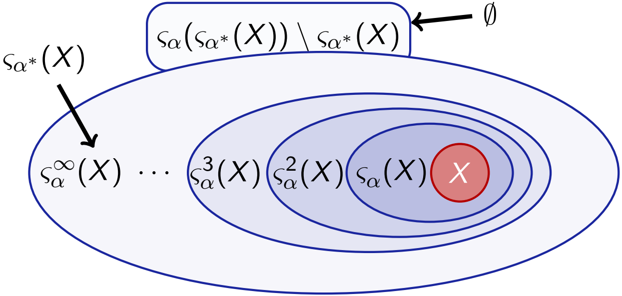

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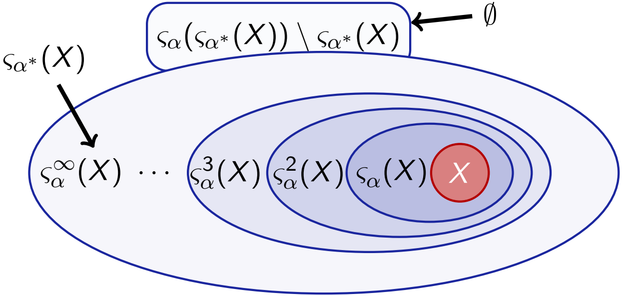

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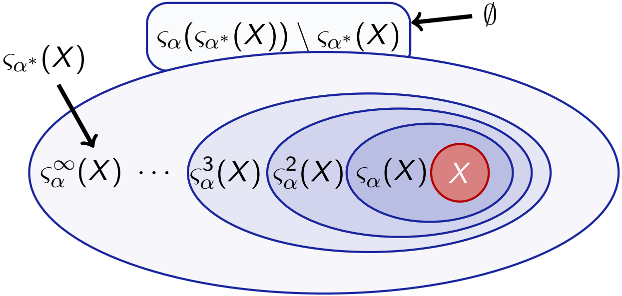

- Which Z is the right one?
- Are there multiple such Z? Does such a Z exist?
- Existence: $Z = \mathcal{S}$ but that's too big and independent of α

Comparing (Pre-)Fixpoints

Lemma ( )

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
$$X \cup \varsigma_\alpha(Z) \subseteq Z$$

are pre-fixpoints, then

Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

$$X \cup \varsigma_\alpha(Y) \subseteq Y$$
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Comparing (Pre-)Fixpoints

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Proof.

$$X \cup \varsigma_\alpha(Y \cap Z) \overset{\text{mon}}{\subseteq} X \cup (\varsigma_\alpha(Y) \cap \varsigma_\alpha(Z)) \overset{\text{above}}{\subseteq} Y \cap Z$$ □

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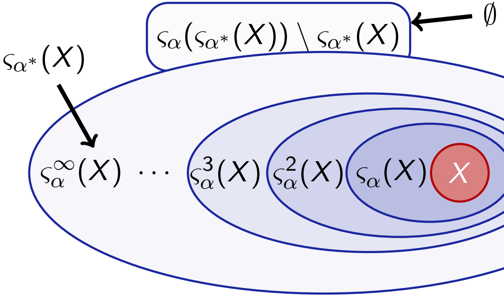
Even: The intersection of *any* family of pre-fixpoints is a pre-fixpoint!
So: repetition semantics is the smallest pre-fixpoint (well-founded)

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$$



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$\varsigma_{\alpha^*}(X)$ intersection of solutions

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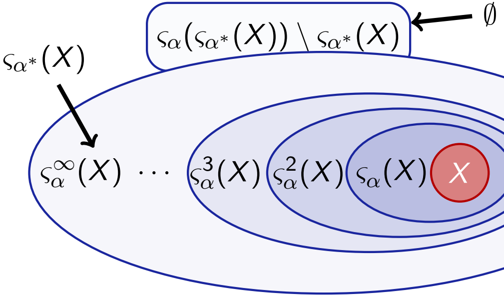

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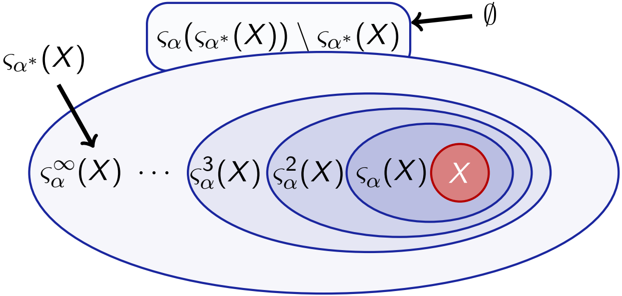

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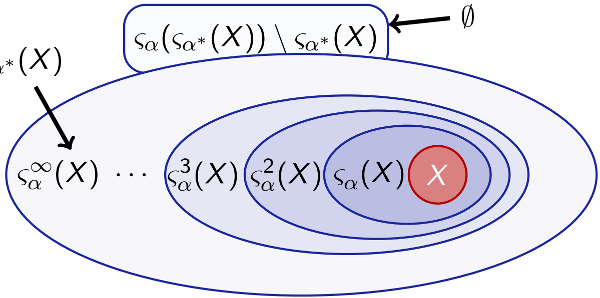

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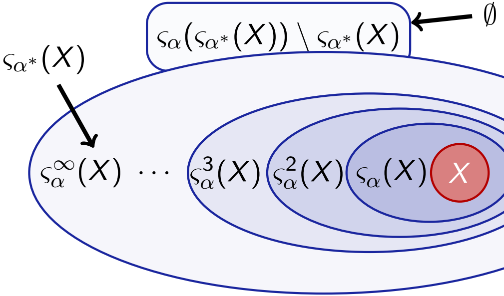

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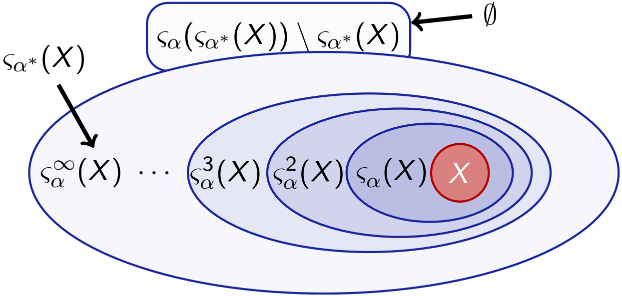

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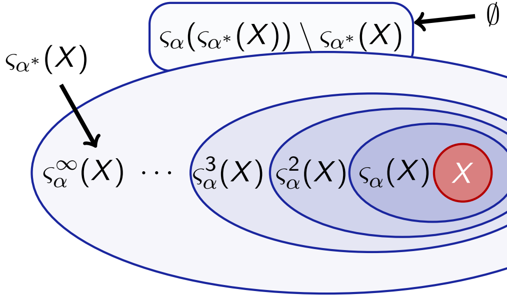

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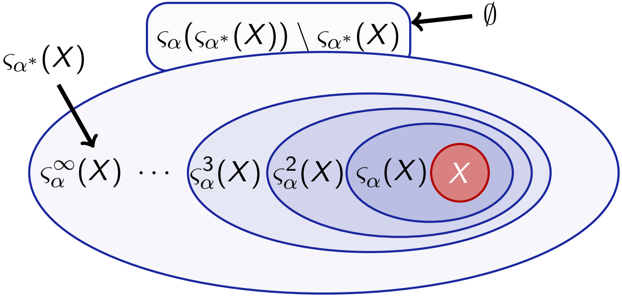
Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) = Z\} = \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X)$ by Knaster-Tarski



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Outline

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathcal{S} : \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x)\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcup_{\kappa<\infty} \varsigma_\alpha^\kappa(X) \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} : \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

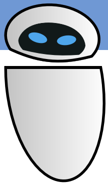
Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathcal{S}) \to \wp(\mathcal{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathcal{S} : \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x)\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathcal{S})$

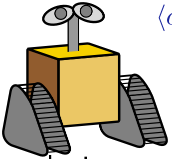
$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathcal{S} : \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

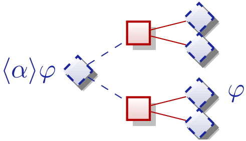
Summary



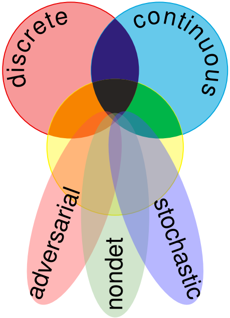
differential game logic

dGL = GL + HG = dL + ^{d}





- Semantics for differential game logic
- Simple compositional denotational semantics
- Meaning is a simple function of its pieces
- Outlier: repetition is subtle higher-ordinal iteration
- Better: repetition means least fixpoint



Next chapter

1. Axiomatics
2. How to win and prove hybrid games

André Platzer.
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