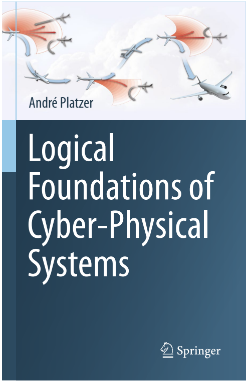
06: Truth & Proof

Logical Foundations of Cyber-Physical Systems


André Platzer

Logical Foundations of Cyber-Physical Systems

Springer

André Platzer

Carnegie Mellon University
Computer Science Department

Outline

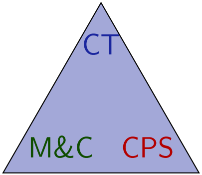
1. Learning Objectives
2. Sequent Calculus
    - Propositional Proof Rules
    - Soundness of Proof Rules
    - Proofs with Dynamics
    - Contextual Equivalence
    - Quantifier Proof Rules
    - A Sequent Proof for Single-hop Bouncing Balls
3. Real Arithmetic
    - Real Quantifier Elimination
    - Instantiating Real-Arithmetic Quantifiers
    - Weakening by Removing Assumptions
    - Abbreviating Terms to Reduce Complexity
    - Substituting Equations into Formulas
    - Creatively Cutting to Transform Questions
4. Summary

Outline

Learning Objectives

Truth & Proof

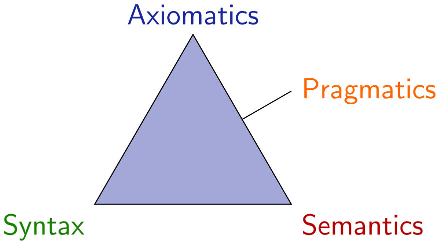
systematic reasoning for CPS
verifying CPS models at scale
pragmatics: how to use axiomatics to justify truth
structure of proofs and their arithmetic



discrete+continuous relation
with evolution domains

analytic skills for CPS

Logical Trinity with Extra Leg



Syntax	defines the notation What problems are we allowed to write down?
Semantics	what carries meaning. What real or mathematical objects does the syntax stand for?
Axiomatics	internalizes semantic relations into universal syntactic transformations.
Pragmatics	how to use axiomatics to justify syntactic rendition of semantical concepts. How to do a proof?

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4. Summary

Sequent Calculus

Definition (Sequent)

$$\Gamma \vdash \Delta$$

has the same meaning as $\bigwedge_{P \in \Gamma} P \to \bigvee_{Q \in \Delta} Q$.
The *antecedent* Γ and *succedent* Δ are finite sets of dL formulas.

Definition (Soundness of sequent calculus proof rules)

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

is *sound* iff validity of all premises implies validity of conclusion:

$$\text{If } \vDash (\Gamma_1 \vdash \Delta_1) \text{ and } \dots \text{ and } \vDash (\Gamma_n \vdash \Delta_n) \text{ then } \vDash (\Gamma \vdash \Delta)$$

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construct proofs up $\uparrow$

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Definition (Soundness of sequent calculus proof rules)

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$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$
$\downarrow$ validity transfers down

is *sound* iff validity of all premises implies validity of conclusion:

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Propositional Proof Rules of Sequent Calculus

$$\wedge\text{L}\ \frac{}{\Gamma, P \wedge Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge L\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta}$$

$\wedge$L: assume conjuncts separately
It successively handles all top-level $\wedge$ in assumptions but not nested in $A \vee (B \wedge C) \vdash C$ which needs rules for other propositional operators

Propositional Proof Rules of Sequent Calculus

$$\wedge\mathrm{R}\ \frac{}{\Gamma \vdash P \wedge Q, \Delta}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

$$\wedge R \ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta}$$

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$\vee$R: split disjunctions in succedent where comma has a disjunctive meaning

Propositional Proof Rules of Sequent Calculus

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$\vee$L: handle disjunctive assumption by one proof for each assumed disjunct

Propositional Proof Rules of Sequent Calculus

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$$\rightarrow R\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$\rightarrow$R: prove implication by assuming LHS when proving RHS

Propositional Proof Rules of Sequent Calculus

$$\wedge R \ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R \ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

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$$\rightarrow\text{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

$\rightarrow$L: assume RHS of an assumed implication after proving its LHS

Propositional Proof Rules of Sequent Calculus

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$\neg$R: prove $\neg P$ by proving contradiction (or Δ options) from assumption P

Propositional Proof Rules of Sequent Calculus

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$\neg$L: assume $\neg P$ by proving its opposite P

Propositional Proof Rules of Sequent Calculus

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$$\wedge L\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg L\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

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Propositional Proof Rules of Sequent Calculus

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id: proof done (marked $*$) when succedent to prove is in antecedent

Propositional Proof Rules of Sequent Calculus

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$$\rightarrow L\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

id: only way to finish a proof (in propositional logic!)

Propositional Proof Rules of Sequent Calculus

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$$\rightarrow\mathrm{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \mathrm{cut}\ \frac{}{\Gamma \vdash \Delta}$$

Propositional Proof Rules of Sequent Calculus

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Propositional Proof Rules of Sequent Calculus

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$$\to\mathrm{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \to Q \vdash \Delta} \qquad \mathrm{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}$$

cut: Show lemma C and then assume lemma C

Propositional Proof Rules of Sequent Calculus

$$\wedge R\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \neg R\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

$$\wedge L\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \neg L\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$$

$$\rightarrow R\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top R\ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

$$\rightarrow L\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}$$

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$\top$R: proof done (marked $*$) when proving trivial *true* (used rarely)

Propositional Proof Rules of Sequent Calculus

$$\land R \; \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \land Q, \Delta} \qquad \lor R \; \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \lor Q, \Delta} \qquad \neg R \; \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$$

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$$\rightarrow R \; \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id} \; \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top R \; \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

$$\rightarrow L \; \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut} \; \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}$$

$\top$R: what rule to use when *true* in antecedent?

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$$\rightarrow\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \mathrm{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top\mathrm{R}\ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

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$\bot$L: proof done (marked $*$) when assuming trivial *false* (used rarely)

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$$\to\mathrm{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \to Q, \Delta} \qquad \mathrm{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top\mathrm{R}\ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

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$\bot$L: what rule to use when *false* in succedent?

Sequent Proof Example (Simple)

$$\frac{}{\vdash v^2 \leq 10 \land b > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq 10)}$$

Sequent Proof Example (Simple)

$$\rightarrow\!R\ \dfrac{\dfrac{}{v^2\leq 10 \land b>0 \vdash b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}}{\vdash v^2\leq 10 \land b>0 \rightarrow b>0 \land (\neg(v\geq 0) \lor v^2\leq 10)}$$

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$$\mathord{\to}\mathrm{R}\;\dfrac{\mathord{\land}\mathrm{R}\;\dfrac{\overline{v^2\le 10 \land b>0 \vdash b>0} \qquad \overline{v^2\le 10 \land b>0 \vdash \lnot(v\ge 0) \lor v^2\le 10}}{v^2\le 10 \land b>0 \vdash b>0 \land (\lnot(v\ge 0) \lor v^2\le 10)}}{\vdash v^2\le 10 \land b>0 \to b>0 \land (\lnot(v\ge 0) \lor v^2\le 10)}$$

Sequent Proof Example (Simple)

$$
{\to}\mathrm{R}\,\dfrac{{\land}\mathrm{R}\,\dfrac{{\land}\mathrm{L}\,\dfrac{\dfrac{}{v^2\leq 10, b>0 \vdash b>0}}{v^2\leq 10 \land b>0 \vdash b>0} \qquad \dfrac{}{v^2\leq 10 \land b>0 \vdash \lnot(v\geq 0) \lor v^2\leq 10}}{v^2\leq 10 \land b>0 \vdash b>0 \land (\lnot(v\geq 0) \lor v^2\leq 10)}}{\vdash v^2\leq 10 \land b>0 \to b>0 \land (\lnot(v\geq 0) \lor v^2\leq 10)}
$$

Sequent Proof Example (Simple)

$$
\text{→R}\,\dfrac{\text{∧R}\,\dfrac{\text{∧L}\,\dfrac{\text{id}\,\dfrac{*}{v^2{\le}10,\, {\color{red}b{>}0} \vdash {\color{red}b{>}0}}}{v^2{\le}10 \land b{>}0 \vdash b{>}0} \qquad \dfrac{}{v^2{\le}10 \land b{>}0 \vdash \neg(v{\ge}0) \lor v^2{\le}10}}{v^2{\le}10 \land b{>}0 \vdash b{>}0 \land (\neg(v{\ge}0) \lor v^2{\le}10)}}{\vdash v^2{\le}10 \land b{>}0 \to b{>}0 \land (\neg(v{\ge}0) \lor v^2{\le}10)}
$$

Sequent Proof Example (Simple)

$$
{\scriptstyle\to R}\,\dfrac{
  {\scriptstyle\land R}\,\dfrac{
    {\scriptstyle\land L}\,\dfrac{
      {\scriptstyle\text{id}}\,\dfrac{*}{v^2\le 10,\, b>0 \vdash b>0}
    }{v^2\le 10 \land b>0 \vdash b>0}
    \qquad
    {\scriptstyle\lor R}\,\dfrac{
      \dfrac{}{v^2\le 10 \land b>0 \vdash \neg(v\ge 0),\, v^2\le 10}
    }{v^2\le 10 \land b>0 \vdash \neg(v\ge 0) \mathbin{\color{red}\lor} v^2\le 10}
  }{v^2\le 10 \land b>0 \vdash b>0 \land (\neg(v\ge 0) \lor v^2\le 10)}
}{\vdash v^2\le 10 \land b>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le 10)}
$$

Sequent Proof Example (Simple)

$$
\begin{prooftree}
\AxiomC{$*$}
\LeftLabel{id}
\UnaryInfC{$v^2{\le}10, b{>}0 \vdash b{>}0$}
\LeftLabel{$\land$L}
\UnaryInfC{$v^2{\le}10 \land b{>}0 \vdash b{>}0$}
\AxiomC{}
\UnaryInfC{$v^2{\le}10, b{>}0 \vdash \neg(v{\ge}0), v^2{\le}10$}
\LeftLabel{$\land$L}
\UnaryInfC{$v^2{\le}10 \land b{>}0 \vdash \neg(v{\ge}0), v^2{\le}10$}
\LeftLabel{$\lor$R}
\UnaryInfC{$v^2{\le}10 \land b{>}0 \vdash \neg(v{\ge}0) \lor v^2{\le}10$}
\LeftLabel{$\land$R}
\BinaryInfC{$v^2{\le}10 \land b{>}0 \vdash b{>}0 \land (\neg(v{\ge}0) \lor v^2{\le}10)$}
\LeftLabel{$\to$R}
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\end{prooftree}
$$

Sequent Proof Example (Simple)

$$
{\to}\mathrm{R}\,\dfrac{{\wedge}\mathrm{R}\,\dfrac{{\wedge}\mathrm{L}\,\dfrac{\mathrm{id}\,\dfrac{*}{v^2{\le}10, b{>}0 \vdash b{>}0}}{v^2{\le}10 \wedge b{>}0 \vdash b{>}0} \qquad {\vee}\mathrm{R}\,\dfrac{{\wedge}\mathrm{L}\,\dfrac{\mathrm{id}\,\dfrac{*}{{\color{red}v^2{\le}10}, b{>}0 \vdash \neg(v{\ge}0), {\color{red}v^2{\le}10}}}{v^2{\le}10 \wedge b{>}0 \vdash \neg(v{\ge}0), v^2{\le}10}}{v^2{\le}10 \wedge b{>}0 \vdash \neg(v{\ge}0) \vee v^2{\le}10}}{v^2{\le}10 \wedge b{>}0 \vdash b{>}0 \wedge (\neg(v{\ge}0) \vee v^2{\le}10)}}{\vdash v^2{\le}10 \wedge b{>}0 \to b{>}0 \wedge (\neg(v{\ge}0) \vee v^2{\le}10)}
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Soundness of Proof Rules

Lemma

$$\wedge R \quad \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \text{ is sound}$$

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Proof

using $[\![P \wedge Q]\!] = [\![P]\!] \cap [\![Q]\!]$.

WLOG: $\omega \in [\![G]\!]$ for all $G \in \Gamma$ and $\omega \notin [\![D]\!]$ for all $D \in \Delta$ (why?)

By premise: $\omega \in [\![\Gamma \vdash P, \Delta]\!]$ and $\omega \in [\![\Gamma \vdash Q, \Delta]\!]$

By WLOG: $\omega \in [\![P]\!]$ and $\omega \in [\![Q]\!]$

By semantics: $\omega \in [\![P \wedge Q]\!]$

By definition: $\omega \in [\![\Gamma \vdash P \wedge Q, \Delta]\!]$ $\square$

Soundness of dL

Theorem

dL *sequent calculus is sound: every* dL *formula with a proof is valid.*

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1. Sequent proof ends with some proof step:

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

The subproof of each premise $\Gamma_i \vdash \Delta_i$ is smaller, so $\vDash \Gamma_i \vdash \Delta_i$ by IH. All dL proof rules are proved sound, also the one used above, i.e.:

$$\text{If } \vDash (\Gamma_1 \vdash \Delta_1) \text{ and } \dots \text{ and } \vDash (\Gamma_n \vdash \Delta_n) \text{ then } \vDash (\Gamma \vdash \Delta)$$

Thus, $\vDash (\Gamma \vdash \Delta)$. ☐

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Thus, $\vDash (\Gamma \vdash \Delta)$. □

▸ Todo Always make sure *every* axiom and proof rule we adopt is sound!

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Have: Left and right proof rule for all propositional connectives

Need: Left and right proof rule for all top-level operators in all modalities?

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Boring! Already follow from the axiom

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$

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Rules $[\cup]R$,$[\cup]L$ would only apply top-level, not in any other logical context such as

$$[x'' = -g]\_$$

$$[\cup]\ \frac{}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v)}$$

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Contextual Equivalence: substituting equals for equals

$$\text{CER}\ \frac{\Gamma \vdash C(Q), \Delta \quad \vdash P \leftrightarrow Q}{\Gamma \vdash C(P), \Delta} \qquad \text{CEL}\ \frac{\Gamma, C(Q) \vdash \Delta \quad \vdash P \leftrightarrow Q}{\Gamma, C(P) \vdash \Delta}$$

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$$[?x{=}0; v := -cv \cup ?x{\geq}0]B(x,v) \leftrightarrow [?x{=}0; v := -cv]B(x,v) \land [?x{\geq}0]B(x,v)$$

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$$[\cup] \ \frac{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big)}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}$$

Simple Example Proof with Dynamics in Sequent Calculus

$$[;]\ \frac{}{\vdash [a := -b; c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}$$

Simple Example Proof with Dynamics in Sequent Calculus

$$[a := -b; c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big) \leftrightarrow$$
$$[a := -b][c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big) \text{ by } [;]$$

$$\dfrac{\dfrac{}{\vdash [a := -b][c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}{[:=]}}{\vdash [a := -b; c := 10]\big(v^2 \leq 10 \land -a > 0 \rightarrow b > 0 \land (\neg(v \geq 0) \lor v^2 \leq c)\big)}{[;]}$$

Simple Example Proof with Dynamics in Sequent Calculus

$$[a := -b][c := 10]\big(v^2 \le 10 \land -a > 0 \to b > 0 \land (\lnot(v \ge 0) \lor v^2 \le c)\big) \leftrightarrow$$
$$[c := 10]\big(v^2 \le 10 \land -(-b) > 0 \to b > 0 \land (\lnot(v \ge 0) \lor v^2 \le c)\big) \text{ by } [:=]$$

$$\begin{array}{ll}
{[:=]} & \overline{\vdash [c := 10]\big(v^2 \le 10 \land -(\,-\,b) > 0 \to b > 0 \land (\lnot(v \ge 0) \lor v^2 \le c)\big)} \\
{[:=]} & \overline{\vdash [a := -b][c := 10]\big(v^2 \le 10 \land -a > 0 \to b > 0 \land (\lnot(v \ge 0) \lor v^2 \le c)\big)} \\
{[;]} & \vdash [a := -b; c := 10]\big(v^2 \le 10 \land -a > 0 \to b > 0 \land (\lnot(v \ge 0) \lor v^2 \le c)\big)
\end{array}$$

Simple Example Proof with Dynamics in Sequent Calculus

$[c := 10]\big(v^2\le 10 \land -(-b)>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le c)\big) \leftrightarrow$
$v^2\le 10 \land -(-b)>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le 10)$ by $[:=]$

$$\begin{array}{rl}
\hline
 & \vdash v^2\le 10 \land \ -(-b)>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le 10) \\
\hline
[:=] & \vdash [c := 10]\big(v^2\le 10 \land -(\,-b)>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le c)\big) \\
\hline
[:=] & \vdash [a := -b][c := 10]\big(v^2\le 10 \land -a>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le c)\big) \\
\hline
[;] & \vdash [a := -b;\, c := 10]\big(v^2\le 10 \land -a>0 \to b>0 \land (\neg(v\ge 0) \lor v^2\le c)\big)
\end{array}$$

Simple Example Proof with Dynamics in Sequent Calculus

$$
{\scriptstyle [;]}\dfrac{
{\scriptstyle [:=]}\dfrac{
{\scriptstyle [:=]}\dfrac{
\dfrac{
{\scriptstyle \to R}\dfrac{
{\scriptstyle \wedge R}\dfrac{
{\scriptstyle \wedge L}\dfrac{
{\scriptstyle \mathrm{id}}\dfrac{*}{v^2\leq 10,\, b>0 \vdash b>0}
}{v^2\leq 10 \wedge b>0 \vdash b>0}
\qquad
{\scriptstyle \vee R}\dfrac{
{\scriptstyle \wedge L}\dfrac{
{\scriptstyle \mathrm{id}}\dfrac{*}{v^2\leq 10,\, b>0 \vdash \neg(v\geq 0),\, v^2\leq 10}
}{v^2\leq 10 \wedge b>0 \vdash \neg(v\geq 0),\, v^2\leq 10}
}{v^2\leq 10 \wedge b>0 \vdash \neg(v\geq 0) \vee v^2\leq 10}
}{v^2\leq 10 \wedge b>0 \vdash b>0 \wedge (\neg(v\geq 0) \vee v^2\leq 10)}
}{\vdash v^2\leq 10 \wedge b>0 \to b>0 \wedge (\neg(v\geq 0) \vee v^2\leq 10)}
}{\vdash v^2\leq 10 \wedge\ -(-b)>0 \to b>0 \wedge (\neg(v\geq 0) \vee v^2\leq 10)}
}{\vdash [c:=10]\big(v^2\leq 10 \wedge -(-b)>0 \to b>0 \wedge (\neg(v\geq 0) \vee v^2\leq c)\big)}
}{\vdash [a:=-b][c:=10]\big(v^2\leq 10 \wedge -a>0 \to b>0 \wedge (\neg(v\geq 0) \vee v^2\leq c)\big)}
}{\vdash [a:=-b;\, c:=10]\big(v^2\leq 10 \wedge -a>0 \to b>0 \wedge (\neg(v\geq 0) \vee v^2\leq c)\big)}
$$

Simple Example Proof with Dynamics in Sequent Calculus

$$
{\scriptstyle[;]}\dfrac{
  {\scriptstyle[:=]}\dfrac{
    {\scriptstyle[:=]}\dfrac{
      \dfrac{
        {\scriptstyle\to\mathrm{R}}\dfrac{
          {\scriptstyle\land\mathrm{R}}\dfrac{
            {\scriptstyle\land\mathrm{L}}\dfrac{
              {\scriptstyle\mathrm{id}}\dfrac{*}{v^2\le10,\, b>0 \vdash b>0}
            }{v^2\le10 \land b>0 \vdash b>0}
            \qquad
            {\scriptstyle\lor\mathrm{R}}\dfrac{
              {\scriptstyle\land\mathrm{L}}\dfrac{
                {\scriptstyle\mathrm{id}}\dfrac{*}{v^2\le10,\, b>0 \vdash \neg(v\ge0),\, v^2\le10}
              }{v^2\le10 \land b>0 \vdash \neg(v\ge0),\, v^2\le10}
            }{v^2\le10 \land b>0 \vdash \neg(v\ge0) \lor v^2\le10}
          }{v^2\le10 \land b>0 \vdash b>0 \land (\neg(v\ge0) \lor v^2\le10)}
        }{\vdash v^2\le10 \land \textcolor{red}{b>0} \to b>0 \land (\neg(v\ge0) \lor v^2\le10)}
      }{\vdash v^2\le10 \land \textcolor{red}{-(-b)>0} \to b>0 \land (\neg(v\ge0) \lor v^2\le10)}
    }{\vdash [c:=10]\big(v^2\le10 \land -(-b)>0 \to b>0 \land (\neg(v\ge0) \lor v^2\le c)\big)}
  }{\vdash [a:=-b][c:=10]\big(v^2\le10 \land -a>0 \to b>0 \land (\neg(v\ge0) \lor v^2\le c)\big)}
}{\vdash [a:=-b;\, c:=10]\big(v^2\le10 \land -a>0 \to b>0 \land (\neg(v\ge0) \lor v^2\le c)\big)}
$$

Need to reason about real arithmetic

Here: to glue previous propositional proof with this dynamic proof

Quantifier Proof Rules

$$\forall R\ \frac{}{\Gamma \vdash \forall x\, p(x), \Delta}$$

Quantifier Proof Rules

$$\forall\mathrm{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}$$

Quantifier Proof Rules

$$\forall R \ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$\forall$R: show for fresh variable y about which we can't know anything

Quantifier Proof Rules

$$\forall\mathrm{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x))$$

$$\exists\mathrm{R}\ \frac{}{\Gamma \vdash \exists x\, p(x), \Delta}$$

Quantifier Proof Rules

$$\forall\mathrm{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\exists\mathrm{R}\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}$$

Quantifier Proof Rules

$$\forall\text{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\exists\text{R}\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad \text{(arbitrary term } e\text{)}$$

$\exists$R: enough to show for any witness term e

Quantifier Proof Rules

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\forall L\ \frac{}{\Gamma, \forall x\, p(x) \vdash \Delta}$$

$$\exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad \text{(arbitrary term } e\text{)}$$

Quantifier Proof Rules

$$\forall R \; \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\forall L \; \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}$$

$$\exists R \; \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad \text{(arbitrary term } e\text{)}$$

Quantifier Proof Rules

$$\forall R \ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \not\in \Gamma,\Delta,\forall x\, p(x))$$

$$\forall L \ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad (\text{arbitrary term } e)$$

$$\exists R \ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

$\forall$L: even holds for arbitrary term e

Quantifier Proof Rules

$$\forall\mathrm{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\forall\mathrm{L}\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad (\text{arbitrary term } e)$$

$$\exists\mathrm{R}\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

$$\exists\mathrm{L}\ \frac{}{\Gamma, \exists x\, p(x) \vdash \Delta}$$

Quantifier Proof Rules

$$\forall R \quad \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\forall L \quad \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad (\text{arbitrary term } e)$$

$$\exists R \quad \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

$$\exists L \quad \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}$$

Quantifier Proof Rules

$$\forall\mathrm{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\forall\mathrm{L}\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad (\text{arbitrary term } e)$$

$$\exists\mathrm{R}\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

$$\exists\mathrm{L}\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta} \quad (y \notin \Gamma,\Delta,\exists x\, p(x))$$

∃L: assume for fresh variable y about which we can't know anything

Quantifier Proof Rules

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma,\Delta,\forall x\, p(x))$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \quad (\text{arbitrary term } e)$$

$$\exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta} \quad (\text{arbitrary term } e)$$

$$\exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta} \quad (y \notin \Gamma,\Delta,\exists x\, p(x))$$

Important: soundness means that conclusion valid if all premises valid.

A **Sequent** Proof of a Single-hop Bouncing Ball

$$\dfrac{}{\vdash A \rightarrow [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}\ \rightarrow\text{R}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$\begin{array}{rl} {\scriptstyle [;]} & \overline{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)} \\ {\scriptstyle \to\mathrm{R}} & \overline{\quad \vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)} \end{array}$$

$$A \overset{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$
\begin{array}{ll}
 & A \vdash \forall t{\geq}0\, \big((H-\tfrac{g}{2}t^2{=}0 \to B(H-\tfrac{g}{2}t^2, -c(-gt))\big) \land \big(H-\tfrac{g}{2}t^2{\geq}0 \to B(H-\tfrac{g}{2}t^2 \\ \hline
{[:=]} & A \vdash \forall t{\geq}0\, [x := H-\tfrac{g}{2}t^2]\big((x{=}0 \to B(x, -c(-gt))) \land (x{\geq}0 \to B(x, -gt))\big) \\ \hline
{[:=]} & A \vdash \forall t{\geq}0\, [x := H-\tfrac{g}{2}t^2][v := -gt]\big((x{=}0 \to B(x, -cv)) \land (x{\geq}0 \to B(x, v))\big) \\ \hline
{[;]} & A \vdash \forall t{\geq}0\, [x := H-\tfrac{g}{2}t^2; v := -gt]\big((x{=}0 \to B(x, -cv)) \land (x{\geq}0 \to B(x, v))\big) \\ \hline
{[']} & A \vdash [x'' = -g]\big((x = 0 \to B(x, -cv)) \land (x \geq 0 \to B(x, v))\big) \\ \hline
{[:=]} & A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x, v)) \land (x \geq 0 \to B(x, v))\big) \\ \hline
{[?]} & A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big) \\ \hline
{[;]} & A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big) \\ \hline
{[\cup]} & A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v) \\ \hline
{[;]} & A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v) \\ \hline
{\to}\mathrm{R} & \quad \vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)
\end{array}
$$

$$A \overset{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x, v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$[x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v) \leftrightarrow$$
$$[x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \text{ by } [;]$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v)}{}\,[\cup]}{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}\,[;]}{\vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)}\,{\to}\mathrm{R}$$

$$A \overset{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$
$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$
$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$[?x=0;v:=-cv\cup ?x\geq 0]B(x,v)\leftrightarrow$$
$$\big([?x=0;v:=-cv]B(x,v)\land[?x\geq 0]B(x,v)\big)\ \text{by}\ [\cup]$$

$$
\begin{array}{ll}
 & \\ \hline
[;] & A\vdash[x''=-g]\big([?x=0;v:=-cv]B(x,v)\land[?x\geq 0]B(x,v)\big) \\ \hline
[\cup] & A\vdash[x''=-g][?x=0;v:=-cv\cup ?x\geq 0]B(x,v) \\ \hline
[;] & A\vdash[x''=-g;(?x=0;v:=-cv\cup ?x\geq 0)]B(x,v) \\ \hline
\rightarrow\mathrm{R} & \vdash A\rightarrow[x''=-g;(?x=0;v:=-cv\cup ?x\geq 0)]B(x,v)
\end{array}
$$

$$A\overset{\text{def}}{\equiv}0\leq x\land x=H\land v=0\land g>0\land 1\geq c\geq 0$$

$$B(x,v)\overset{\text{def}}{\equiv}0\leq x\land x\leq H$$

$$\{x''=-g\}\overset{\text{def}}{\equiv}\{x'=v,v'=-g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$[?x=0;v:=-cv]B(x,v) \leftrightarrow$$
$$[?x=0][v:=-cv]B(x,v) \text{ by } [;]$$

$$
\begin{array}{rl}
{\scriptstyle [?]} & \overline{A \vdash [x''=-g]\big([?x=0][v:=-cv]B(x,v) \land [?x\geq 0]B(x,v)\big)} \\
{\scriptstyle [;]} & \overline{A \vdash [x''=-g]\big([?x=0;v:=-cv]B(x,v) \land [?x\geq 0]B(x,v)\big)} \\
{\scriptstyle [\cup]} & \overline{A \vdash [x''=-g][?x=0;v:=-cv \cup ?x\geq 0]B(x,v)} \\
{\scriptstyle [;]} & \overline{A \vdash [x''=-g;(?x=0;v:=-cv \cup ?x\geq 0)]B(x,v)} \\
{\scriptstyle \to R} & \overline{\vdash A \to [x''=-g;(?x=0;v:=-cv \cup ?x\geq 0)]B(x,v)}
\end{array}
$$

$$A \overset{\text{def}}{\equiv} 0\leq x \land x=H \land v=0 \land g>0 \land 1\geq c\geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0\leq x \land x\leq H$$

$$\{x''=-g\} \overset{\text{def}}{\equiv} \{x'=v, v'=-g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$[?x=0][v:=-cv]B(x,v) \leftrightarrow$$
$$x=0 \to [v:=-cv]B(x,v) \text{ by } [?]$$

$$
\begin{array}{ll}
\hline
[:=] & A \vdash [x''=-g]\big((x=0 \to [v:=-cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
[?] & A \vdash [x''=-g]\big([?x=0][v:=-cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[;] & A \vdash [x''=-g]\big([?x=0; v:=-cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[\cup] & A \vdash [x''=-g][?x=0; v:=-cv \cup ?x \geq 0]B(x,v) \\
\hline
[;] & A \vdash [x''=-g; (?x=0; v:=-cv \cup ?x \geq 0)]B(x,v) \\
\hline
\to\mathrm{R} & \vdash A \to [x''=-g; (?x=0; v:=-cv \cup ?x \geq 0)]B(x,v)
\end{array}
$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x=H \land v=0 \land g>0 \land 1 \geq c \geq 0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x''=-g\} \overset{\text{def}}{\equiv} \{x'=v, v'=-g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$[v := -cv]B(x, v) \leftrightarrow$
$x = 0 \to B(x, -cv)$ by $[:=]$

$$\begin{array}{rl}
['] & \overline{A \vdash [x'' = -g]\big((x = 0 \to B(x, -cv)) \land (x \geq 0 \to B(x, v))\big)} \\
[:=] & \overline{A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x, v)) \land (x \geq 0 \to B(x, v))\big)} \\
[?] & \overline{A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big)} \\
[;] & \overline{A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x, v) \land [?x \geq 0]B(x, v)\big)} \\
[\cup] & \overline{A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x, v)} \\
[;] & \overline{A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)} \\
\to\mathrm{R} & \overline{\vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x, v)}
\end{array}$$

$$A \overset{\text{def}}{\equiv} 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x, v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$[{}'] \quad [x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\, [x := y(t)]p(x)$$

$$\begin{array}{ll}
\hline
[;] & A \vdash \forall t{\geq}0\, [x := H{-}\frac{g}{2}t^2; v := -gt]\big((x{=}0{\to}B(x,-cv)) \land (x{\geq}0{\to}B(x,v))\big) \\
\hline
[{}'] & A \vdash [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
[:=] & A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
[?] & A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[;] & A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[\cup] & A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \\
\hline
[;] & A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v) \\
\hline
{\to}\mathrm{R} & \vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}$$

$$A \overset{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$
\begin{array}{ll}
\hline
\text{[:=]} & A \vdash \forall t{\geq}0\,[x := H{-}\frac{g}{2}t^2][v := -gt]\big((x{=}0{\to}B(x,-cv)) \land (x{\geq}0{\to}B(x,v))\big) \\
\hline
\text{[;]} & A \vdash \forall t{\geq}0\,[x := H{-}\frac{g}{2}t^2;\, v := -gt]\big((x{=}0{\to}B(x,-cv)) \land (x{\geq}0{\to}B(x,v))\big) \\
\hline
\text{[']} & A \vdash [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
\text{[:=]} & A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
\text{[?]} & A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
\text{[;]} & A \vdash [x'' = -g]\big([?x = 0;\, v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
\text{[}\cup\text{]} & A \vdash [x'' = -g][?x = 0;\, v := -cv \cup ?x \geq 0]B(x,v) \\
\hline
\text{[;]} & A \vdash [x'' = -g;\, (?x = 0;\, v := -cv \cup ?x \geq 0)]B(x,v) \\
\hline
\to\text{R} & \quad \vdash A \to [x'' = -g;\, (?x = 0;\, v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}
$$

$$A \overset{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$\begin{array}{ll}
\hline
[:=] & A \vdash \forall t{\geq}0\, [x := H{-}\frac{g}{2}t^2]\big((x{=}0{\to}B(x,-c(-gt))) \land (x{\geq}0{\to}B(x,-gt))\big) \\
\hline
[:=] & A \vdash \forall t{\geq}0\, [x := H{-}\frac{g}{2}t^2][v := -gt]\big((x{=}0{\to}B(x,-cv)) \land (x{\geq}0{\to}B(x,v))\big) \\
\hline
[;] & A \vdash \forall t{\geq}0\, [x := H{-}\frac{g}{2}t^2; v := -gt]\big((x{=}0{\to}B(x,-cv)) \land (x{\geq}0{\to}B(x,v))\big) \\
\hline
['] & A \vdash [x'' = -g]\big((x = 0 \to B(x,-cv)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
[:=] & A \vdash [x'' = -g]\big((x = 0 \to [v := -cv]B(x,v)) \land (x \geq 0 \to B(x,v))\big) \\
\hline
[?] & A \vdash [x'' = -g]\big([?x = 0][v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[;] & A \vdash [x'' = -g]\big([?x = 0; v := -cv]B(x,v) \land [?x \geq 0]B(x,v)\big) \\
\hline
[\cup] & A \vdash [x'' = -g][?x = 0; v := -cv \cup ?x \geq 0]B(x,v) \\
\hline
[;] & A \vdash [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v) \\
\hline
{\to}\mathrm{R} & \quad \vdash A \to [x'' = -g; (?x = 0; v := -cv \cup ?x \geq 0)]B(x,v)
\end{array}$$

$$A \overset{\text{def}}{\equiv} 0{\leq}x \land x{=}H \land v{=}0 \land g{>}0 \land 1{\geq}c{\geq}0$$

$$B(x,v) \overset{\text{def}}{\equiv} 0 \leq x \land x \leq H$$

$$\{x'' = -g\} \overset{\text{def}}{\equiv} \{x' = v, v' = -g\}$$

A **Sequent** Proof of a Single-hop Bouncing Ball

$$\begin{array}{rl}
 & A \vdash \forall t{\geq}0\, \big((H{-}\tfrac{g}{2}t^2{=}0{\to}B(H{-}\tfrac{g}{2}t^2, -c(-gt))) \land (H{-}\tfrac{g}{2}t^2{\geq}0{\to}B(H{-}\tfrac{g}{2}t^2 \\ \hline
[:=] & A \vdash \forall t{\geq}0\, [x := H{-}\tfrac{g}{2}t^2]\big((x{=}0{\to}B(x, -c(-gt))) \land (x{\geq}0{\to}B(x, -gt))\big) \\ \hline
[:=] & A \vdash \forall t{\geq}0\, [x := H{-}\tfrac{g}{2}t^2][v := -gt]\big((x{=}0{\to}B(x, -cv)) \land (x{\geq}0{\to}B(x, v))\big) \\ \hline
[;] & A \vdash \forall t{\geq}0\, [x := H{-}\tfrac{g}{2}t^2; v := -gt]\big((x{=}0{\to}B(x, -cv)) \land (x{\geq}0{\to}B(x, v))\big) \\ \hline
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A **Sequent** Proof of a Single-hop Bouncing Ball

$$
\begin{array}{ll}
 & {\color{red} A \vdash \forall t{\geq}0\, \big((H-\frac{g}{2}t^2=0 \to B(H-\frac{g}{2}t^2, -c(-gt))) \land (H-\frac{g}{2}t^2\geq0 \to B(H-\frac{g}{2}t^2} \\ \hline
[:=] & A \vdash \forall t{\geq}0\, [x:=H-\frac{g}{2}t^2]\big((x{=}0\to B(x,-c(-gt))) \land (x{\geq}0\to B(x,-gt))\big) \\ \hline
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Outline

1. Learning Objectives

2. Sequent Calculus
   - Propositional Proof Rules
   - Soundness of Proof Rules
   - Proofs with Dynamics
   - Contextual Equivalence
   - Quantifier Proof Rules
   - A Sequent Proof for Single-hop Bouncing Balls

3. Real Arithmetic
   - Real Quantifier Elimination
   - Instantiating Real-Arithmetic Quantifiers
   - Weakening by Removing Assumptions
   - Abbreviating Terms to Reduce Complexity
   - Substituting Equations into Formulas
   - Creatively Cutting to Transform Questions

4. Summary

Real Arithmetic

Lemma ($\mathbb{R}$ real arithmetic)

$\text{FOL}_{\mathbb{R}}$ *decidable, so side condition implementable:*

$$\mathbb{R}\ \frac{}{\Gamma \vdash \Delta} \qquad (\textit{if } \bigwedge_{P\in\Gamma} P \rightarrow \bigvee_{Q\in\Delta} Q \textit{ is valid in } \text{FOL}_{\mathbb{R}})$$

$$\mathbb{R}\overline{a > 0, b > 0 \vdash y \geq 0 \rightarrow ax^2 + by \geq 0} \qquad \mathbb{R}\overline{x^2 > 0 \vdash x > 0}$$

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Theorem (Tarski's quantifier elimination)

FOL$_{\mathbb{R}}$ *admits quantifier elimination: there is an algorithm that computes a quantifier-free formula* QE(P)*, for each first-order real arithmetic formula* P*, that is equivalent, i.e.,* $P \leftrightarrow \mathrm{QE}(P)$ *is valid.*

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What if there are no quantifiers?

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What if there are no quantifiers? Universal closure with i$\forall$ $\dfrac{\Gamma \vdash \forall x\, P, \Delta}{\Gamma \vdash P, \Delta}$

Quantifier Elimination After Universal Closure

$$\forall\mathrm{R}\ \frac{}{\vdash \forall d\, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\big)}$$

Quantifier Elimination After Universal Closure

$$\forall R \frac{}{\vdash \forall d\, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\big)}$$

Not a $\mathrm{FOL}_{\mathbb{R}}$ formula so Tarski's quantifier elimination not applicable.

Quantifier Elimination After Universal Closure

$$\begin{array}{rl} [\cup] & \overline{\vdash d \geq -x \to [x := 0 \cup x := x + d]\, x \geq 0} \\ \forall\mathrm{R} & \overline{\vdash \forall d \left(d \geq -x \to [x := 0 \cup x := x + d]\, x \geq 0\right)} \end{array}$$

Quantifier Elimination After Universal Closure

$$
\forall\text{R}\ \dfrac{[\cup]\ \dfrac{[:=]\ \dfrac{}{\vdash d \geq -x \to [x := 0]\, x \geq 0 \land [x := x + d]\, x \geq 0}}{\vdash d \geq -x \to [x := 0 \cup x := x + d]\, x \geq 0}}{\vdash \forall d\, \big(d \geq -x \to [x := 0 \cup x := x + d]\, x \geq 0\big)}
$$

Quantifier Elimination After Universal Closure

$$\begin{array}{rl}
 & \\ \cline{2-2}
{\scriptstyle [:=]} & \vdash d \geq -x \rightarrow 0 \geq 0 \land [x := x + d]\, x \geq 0 \\ \cline{2-2}
{\scriptstyle [:=]} & \vdash d \geq -x \rightarrow [x := 0]\, x \geq 0 \land [x := x + d]\, x \geq 0 \\ \cline{2-2}
{\scriptstyle [\cup]} & \vdash d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0 \\ \cline{2-2}
{\scriptstyle \forall\mathrm{R}} & \vdash \forall d\, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\big)
\end{array}$$

Quantifier Elimination After Universal Closure

$$\begin{array}{ll}
\mathrm{i}\forall & \overline{\vdash d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0} \\
[:=] & \overline{\vdash d \geq -x \rightarrow 0 \geq 0 \land [x := x + d]\, x \geq 0} \\
[:=] & \overline{\vdash d \geq -x \rightarrow [x := 0]\, x \geq 0 \land [x := x + d]\, x \geq 0} \\
[\cup] & \overline{\vdash d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0} \\
\forall\mathrm{R} & \overline{\vdash \forall d \left(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\right)}
\end{array}$$

Quantifier Elimination After Universal Closure

$$
\begin{array}{rl}
\text{i}\forall & \overline{\vdash \forall {\color{red}d}\, \big(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\big)} \\
\text{i}\forall & \overline{\vdash d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0} \\
[:=] & \overline{\vdash d \geq -x \rightarrow 0 \geq 0 \land [x := x + d]\, x \geq 0} \\
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[\cup] & \overline{\vdash d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0} \\
\forall\text{R} & \overline{\vdash \forall d\, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\big)}
\end{array}
$$

Quantifier Elimination After Universal Closure

$$
\begin{array}{rl}
\cline{2-2}
\mathbb{R} & \vdash \forall x\,\forall d\,\big(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\big) \\ \cline{2-2}
\mathrm{i}\forall & \vdash \forall d\,\big(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\big) \\ \cline{2-2}
\mathrm{i}\forall & \vdash d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0 \\ \cline{2-2}
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[:=] & \vdash d \geq -x \rightarrow [x := 0]\,x \geq 0 \land [x := x + d]\,x \geq 0 \\ \cline{2-2}
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\forall\mathrm{R} & \vdash \forall d\,\big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\,x \geq 0\big)
\end{array}
$$

Quantifier Elimination After Universal Closure

$$\begin{array}{rl}
 & * \\ \cline{2-2}
\mathbb{R} & \vdash \forall x\, \forall d\, \big(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\big) \\ \cline{2-2}
\mathrm{i}\forall & \vdash \forall d\, \big(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\big) \\ \cline{2-2}
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[:=] & \vdash d \geq -x \rightarrow 0 \geq 0 \land [x := x + d]\, x \geq 0 \\ \cline{2-2}
[:=] & \vdash d \geq -x \rightarrow [x := 0]\, x \geq 0 \land [x := x + d]\, x \geq 0 \\ \cline{2-2}
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\forall\mathrm{R} & \vdash \forall d\, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\big)
\end{array}$$

Quantifier Elimination After Universal Closure

$$
\begin{array}{rl}
 & * \\ \cline{2-2}
\mathbb{R} & \vdash \forall x\,\forall d\,\big(d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0\big) \\ \cline{2-2}
\mathrm{i}\forall & \vdash \forall d\,\big(d \geq -x \rightarrow 0 \geq 0 \land x+d \geq 0\big) \\ \cline{2-2}
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[:=] & \vdash d \geq -x \rightarrow 0 \geq 0 \land [x:=x+d]\,x \geq 0 \\ \cline{2-2}
[:=] & \vdash d \geq -x \rightarrow [x:=0]\,x \geq 0 \land [x:=x+d]\,x \geq 0 \\ \cline{2-2}
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\forall\mathrm{R} & \vdash \forall d\,\big(d \geq -x \rightarrow [x:=0 \cup x:=x+d]\,x \geq 0\big)
\end{array}
$$

We could also leave $\forall d$ alone and use axioms in the middle of the formula.

Quantifier Elimination After Universal Closure

$$
\begin{array}{rl}
 & \ast \\ \hline
\mathbb{R} & \vdash \forall x\, \forall d\, \big(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\big) \\ \hline
\mathrm{i}\forall & \vdash \forall d\, \big(d \geq -x \rightarrow 0 \geq 0 \land x + d \geq 0\big) \\ \hline
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[\cup] & \vdash d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0 \\ \hline
\forall\mathrm{R} & \vdash \forall d\, \big(d \geq -x \rightarrow [x := 0 \cup x := x + d]\, x \geq 0\big)
\end{array}
$$

Already use rule $\mathbb{R}$ for valid $\mathrm{FOL}_{\mathbb{R}}$ formulas with free variables before i$\forall$

Instantiating Real-Arithmetic Quantifiers

$$\forall\text{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(\dots) \quad \exists\text{R}\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}(\dots)$$

$$\forall\text{L}\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}(\dots) \quad \exists\text{L}\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(\dots)$$

$$\frac{}{\Gamma \vdash [x' = f(x) \,\&\, q(x)]P}$$

Instantiating Real-Arithmetic Quantifiers

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(\dots) \quad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}(\dots)$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}(\dots) \quad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(\dots)$$

$$[']\ \frac{\overline{\Gamma \vdash \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, q(y(s))) \rightarrow [x := y(t)]P\big)}}{\Gamma \vdash [x' = f(x)\ \&\ q(x)]P}$$

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$$

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$$

$$
[']\ \dfrac{\forall\text{R}\ \dfrac{\rightarrow\text{R}\ \dfrac{\dfrac{}{\Gamma, t{\geq}0 \vdash (\forall 0{\leq}s{\leq}t\, q(y(s)))\rightarrow[x := y(t)]P}}{\Gamma \vdash t{\geq}0\rightarrow\big((\forall 0{\leq}s{\leq}t\, Q(y(s)))\rightarrow[x := y(t)]P\big)}}{\Gamma \vdash \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, q(y(s)))\rightarrow[x := y(t)]P\big)}}{\Gamma \vdash [x' = f(x)\,\&\, q(x)]P}
$$

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$$
\begin{array}{rc}
\hline
 & \Gamma, t{\geq}0, 0{\leq}t{\leq}t{\to}q(y(t)) \vdash [x := y(t)]P \\
\hline
\forall L & \Gamma, t{\geq}0, \forall 0{\leq}s{\leq}t\, q(y(s)) \vdash [x := y(t)]P \\
\hline
\to R & \Gamma, t{\geq}0 \vdash (\forall 0{\leq}s{\leq}t\, q(y(s))){\to}[x := y(t)]P \\
\hline
\to R & \Gamma \vdash t{\geq}0{\to}\big((\forall 0{\leq}s{\leq}t\, Q(y(s))){\to}[x := y(t)]P\big) \\
\hline
\forall R & \Gamma \vdash \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, q(y(s))){\to}[x := y(t)]P\big) \\
\hline
['] & \Gamma \vdash [x' = f(x)\,\&\, q(x)]P
\end{array}
$$

Instantiating Real-Arithmetic Quantifiers

$$\forall\text{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(\dots) \qquad \exists\text{R}\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}(\dots)$$

$$\forall\text{L}\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}(\dots) \qquad \exists\text{L}\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(\dots)$$

$$
\begin{array}{rc}
 & \overline{t\geq0 \vdash 0\leq t\leq t, [x := y(t)]P} \qquad \overline{\Gamma, t\geq0, q(y(t)) \vdash [x := y(t)]P} \\
\hline
\to\text{L} & \Gamma, t\geq0, 0\leq t\leq t \to q(y(t)) \vdash [x := y(t)]P \\
\hline
\forall\text{L} & \Gamma, t\geq0, \forall 0\leq s\leq t\, q(y(s)) \vdash [x := y(t)]P \\
\hline
\to\text{R} & \Gamma, t\geq0 \vdash (\forall 0\leq s\leq t\, q(y(s))) \to [x := y(t)]P \\
\hline
\to\text{R} & \Gamma \vdash t\geq0 \to \big((\forall 0\leq s\leq t\, Q(y(s))) \to [x := y(t)]P\big) \\
\hline
\forall\text{R} & \Gamma \vdash \forall t\geq0\, \big((\forall 0\leq s\leq t\, q(y(s))) \to [x := y(t)]P\big) \\
\hline
[']\ & \Gamma \vdash [x' = f(x)\,\&\, q(x)]P
\end{array}
$$

Instantiating Real-Arithmetic Quantifiers

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(\ldots) \qquad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}(\ldots)$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}(\ldots) \qquad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(\ldots)$$

$$[']\ \dfrac{\forall R\ \dfrac{{\to}R\ \dfrac{{\to}R\ \dfrac{\forall L\ \dfrac{{\to}L\ \dfrac{\mathbb{R}\ \dfrac{*}{t\geq 0 \vdash 0\leq t\leq t, [x := y(t)]P} \qquad \dfrac{}{\Gamma, t\geq 0, q(y(t)) \vdash [x := y(t)]P}}{\Gamma, t\geq 0, 0\leq t\leq t \to q(y(t) \vdash [x := y(t)]P}}{\Gamma, t\geq 0, \forall 0\leq s\leq t\, q(y(s)) \vdash [x := y(t)]P}}{\Gamma, t\geq 0 \vdash (\forall 0\leq s\leq t\, q(y(s))) \to [x := y(t)]P}}{\Gamma \vdash t\geq 0 \to \big((\forall 0\leq s\leq t\, Q(y(s))) \to [x := y(t)]P\big)}}{\Gamma \vdash \forall t\geq 0\, \big((\forall 0\leq s\leq t\, q(y(s))) \to [x := y(t)]P\big)}}{\Gamma \vdash [x' = f(x)\, \&\, q(x)]P}$$

Instantiating Real-Arithmetic Quantifiers

$$
\forall R\ \dfrac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(\ldots) \qquad \exists R\ \dfrac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}(\ldots)
$$

$$
\forall L\ \dfrac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}(\ldots) \qquad \exists L\ \dfrac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(\ldots)
$$

$$
[{}'] \dfrac{\forall R \dfrac{\to R \dfrac{\to R \dfrac{\forall L \dfrac{\to L \dfrac{\mathbb{R} \dfrac{*}{t{\geq}0 \vdash 0{\leq}t{\leq}t, [x := y(t)]P} \qquad \dfrac{\ldots}{\Gamma, t{\geq}0, q(y(t)) \vdash [x := y(t)]P}}{\Gamma, t{\geq}0, 0{\leq}t{\leq}t \to q(y(t)) \vdash [x := y(t)]P}}{\Gamma, t{\geq}0, \forall 0{\leq}s{\leq}t\, q(y(s)) \vdash [x := y(t)]P}}{\Gamma, t{\geq}0 \vdash (\forall 0{\leq}s{\leq}t\, q(y(s))) \to [x := y(t)]P}}{\Gamma \vdash t{\geq}0 \to \big((\forall 0{\leq}s{\leq}t\, Q(y(s))) \to [x := y(t)]P\big)}}{\Gamma \vdash \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, q(y(s))) \to [x := y(t)]P\big)}}{\Gamma \vdash [x' = f(x) \,\&\, q(x)]P}
$$

Instantiating Real-Arithmetic Quantifiers

Derived Rule $$\frac{\Gamma, t\geq0, q(y(t)) \vdash [x:=y(t)]P}{\Gamma \vdash [x'=f(x)\,\&\,q(x)]P} \qquad (y'(t)=f(y))$$

$$
{\scriptstyle [']}\dfrac{{\scriptstyle \forall R}\dfrac{{\scriptstyle \to R}\dfrac{{\scriptstyle \to R}\dfrac{{\scriptstyle \forall L}\dfrac{{\scriptstyle \to L}\dfrac{{\scriptstyle \mathbb{R}}\dfrac{*}{t\geq0 \vdash 0\leq t\leq t, [x:=y(t)]P} \qquad \dfrac{\dots}{\Gamma, t\geq0, q(y(t)) \vdash [x:=y(t)]P}}{\Gamma, t\geq0, 0\leq t\leq t \to q(y(t)) \vdash [x:=y(t)]P}}{\Gamma, t\geq0, \forall 0\leq s\leq t\, q(y(s)) \vdash [x:=y(t)]P}}{\Gamma, t\geq0 \vdash (\forall 0\leq s\leq t\, q(y(s))) \to [x:=y(t)]P}}{\Gamma \vdash t\geq0 \to \big((\forall 0\leq s\leq t\, Q(y(s))) \to [x:=y(t)]P\big)}}{\Gamma \vdash \forall t\geq0\, \big((\forall 0\leq s\leq t\, q(y(s))) \to [x:=y(t)]P\big)}}{\Gamma \vdash [x'=f(x)\,\&\,q(x)]P}
$$

Derived rule: rule that can be proved using other proof rules.

Weakening by Removing Assumptions

$$\text{WR}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash P, \Delta}$$

$$\text{WL}\ \frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}$$

$$\text{WL}\frac{r\geq 0 \vdash 0\leq r\leq r}{A, r\geq 0 \vdash 0\leq r\leq r}$$

Throw big arithmetic distraction A away by weakening since the proof is independent of formula A.

Occam's assumption razor

Think how hard it would be to prove a theorem with all the facts in all books of mathematics as assumptions.
Compared to a proof from just the two facts that matter.

Abbreviating Terms to Reduce Complexity

$$a\geq0, t\geq0, 0 \leq \underbrace{\frac{a}{2}t^2 + vt + x}_{z}, \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq d, d\leq8 \vdash \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq 8$$

Abbreviate fancy term $\frac{a}{2}t^2 + vt + x$ by new variable z makes it easy:

$$a \geq 0, t \geq 0, 0 \leq z, z \leq d, d \leq 8 \vdash z \leq 8$$

Abbreviating Terms to Reduce Complexity

$$a{\geq}0, t{\geq}0, 0 \leq \underbrace{\frac{a}{2}t^2 + vt + x}_{z}, \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq d, d{\leq}8 \vdash \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq 8$$

Abbreviate fancy term $\frac{a}{2}t^2 + vt + x$ by new variable z makes it easy:

$$a \geq 0, t \geq 0, 0 \leq z, z \leq d, d \leq 8 \vdash z \leq 8$$

Proof rules introducing such new variables will be studied in ▸ Chapter 12

Abbreviating Terms to Reduce Complexity

$$a{\geq}0, t{\geq}0, 0 \leq \underbrace{\frac{a}{2}t^2 + vt + x}_{z}, \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq d, d{\leq}8 \vdash \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq 8$$

Abbreviate fancy term $\frac{a}{2}t^2 + vt + x$ by new variable z makes it easy:

$$a \geq 0, t \geq 0, 0 \leq z, z \leq d, d \leq 8 \vdash z \leq 8$$

Proof rules introducing such new variables will be studied in ‣ Chapter 12
Inverse of a derived rule that turns assignments into equations:

$$[:=]_= \quad \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}$$

Abbreviating Terms to Reduce Complexity

$$a{\geq}0, t{\geq}0, 0 \leq \underbrace{\frac{a}{2}t^2 + vt + x}_{z}, \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq d, d{\leq}8 \vdash \underbrace{\frac{a}{2}t^2 + vt + x}_{z} \leq 8$$

Abbreviate fancy term $\frac{a}{2}t^2 + vt + x$ by new variable z makes it easy:

$$a \geq 0, t \geq 0, 0 \leq z, z \leq d, d \leq 8 \vdash z \leq 8$$

Proof rules introducing such new variables will be studied in ▸ Chapter 12

Inverse of a derived rule that turns assignments into equations:

$$[:=]_= \quad \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta} \quad (y \text{ new})$$

Creatively Cutting to Transform Questions

$$=\!R\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$=\!L\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\dfrac{\dfrac{\dfrac{}{(x-y)^2\leq 0, p(y) \vdash p(x)}\,\text{cut}}{(x-y)^2\leq 0 \land p(y) \vdash p(x)}\,\land L}{\vdash (x-y)^2\leq 0 \land p(y) \rightarrow p(x)}\,\rightarrow R$$

Creatively Cutting to Transform Questions

$$=\!R\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$=\!L\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\to\!R\ \dfrac{\land L\ \dfrac{\text{cut}\ \dfrac{\text{WL}\ \dfrac{}{(x-y)^2 \leq 0, p(y) \vdash x = y, p(x)} \qquad \text{WL}\ \dfrac{}{(x-y)^2 \leq 0, p(y), x = y \vdash p(x)}}{(x-y)^2 \leq 0, p(y) \vdash p(x)}}{(x-y)^2 \leq 0 \land p(y) \vdash p(x)}}{\vdash (x-y)^2 \leq 0 \land p(y) \to p(x)}$$

Creatively Cutting to Transform Questions

$$\text{=R}\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$\text{=L}\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\text{→R}\,\dfrac{\text{∧L}\,\dfrac{\text{cut}\,\dfrac{\text{WL}\,\dfrac{\text{WR}\,\dfrac{\mathbb{R}\,\dfrac{*}{(x-y)^2\le0 \vdash x = y}}{(x-y)^2\le0 \vdash x = y, p(x)}}{(x-y)^2\le0, p(y) \vdash x = y, p(x)} \qquad \text{WL}\,\dfrac{}{(x-y)^2\le0, p(y), x = y \vdash p(x)}}{(x-y)^2\le0, p(y) \vdash p(x)}}{(x-y)^2\le0 \land p(y) \vdash p(x)}}{\vdash (x-y)^2\le0 \land p(y) \to p(x)}$$

Creatively Cutting to Transform Questions

$$\text{=R}\ \dfrac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$\text{=L}\ \dfrac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\to\text{R}\ \dfrac{\land\text{L}\ \dfrac{\text{cut}\ \dfrac{\text{WL}\ \dfrac{\text{WR}\ \dfrac{\mathbb{R}\ \dfrac{*}{(x-y)^2 \le 0 \vdash x = y}}{(x-y)^2 \le 0 \vdash x = y, p(x)}}{(x-y)^2 \le 0, p(y) \vdash x = y, p(x)} \qquad \text{WL}\ \dfrac{\text{=R}\ \dfrac{}{p(y), x = y \vdash p(x)}}{(x-y)^2 \le 0, p(y), x = y \vdash p(x)}}{(x-y)^2 \le 0, p(y) \vdash p(x)}}{(x-y)^2 \le 0 \land p(y) \vdash p(x)}}{\vdash (x-y)^2 \le 0 \land p(y) \to p(x)}$$

Creatively Cutting to Transform Questions

$$\mathrm{{=}R}\ \dfrac{\Gamma, x=e \vdash p(e), \Delta}{\Gamma, x=e \vdash p(x), \Delta}$$

$$\mathrm{{=}L}\ \dfrac{\Gamma, x=e, p(e) \vdash \Delta}{\Gamma, x=e, p(x) \vdash \Delta}$$

$$\mathrm{{\to}R}\ \dfrac{\land\mathrm{L}\ \dfrac{\mathrm{cut}\ \dfrac{\mathrm{WL}\ \dfrac{\mathrm{WR}\ \dfrac{\mathbb{R}\ \dfrac{*}{(x-y)^2\le 0 \vdash x=y}}{(x-y)^2\le 0 \vdash x=y, p(x)}}{(x-y)^2\le 0, p(y) \vdash x=y, p(x)} \qquad \mathrm{WL}\ \dfrac{\mathrm{{=}R}\ \dfrac{\mathrm{id}\ \dfrac{}{p(y), x=y \vdash p(y)}}{p(y), x=y \vdash p(x)}}{(x-y)^2\le 0, p(y), x=y \vdash p(x)}}{(x-y)^2\le 0, p(y) \vdash p(x)}}{(x-y)^2\le 0 \land p(y) \vdash p(x)}}{\vdash (x-y)^2\le 0 \land p(y) \to p(x)}$$

Creatively Cutting to Transform Questions

$$=\!\mathrm{R}\ \dfrac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$$

$$=\!\mathrm{L}\ \dfrac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$

$$\rightarrow\!\mathrm{R}\ \dfrac{\wedge\mathrm{L}\ \dfrac{\mathrm{cut}\ \dfrac{\mathrm{WL}\ \dfrac{\mathrm{WR}\ \dfrac{\mathbb{R}\ \dfrac{*}{(x-y)^2 \leq 0 \vdash x = y}}{(x-y)^2 \leq 0 \vdash x = y, p(x)}}{(x-y)^2 \leq 0, p(y) \vdash x = y, p(x)} \qquad \mathrm{WL}\ \dfrac{=\!\mathrm{R}\ \dfrac{\mathrm{id}\ \dfrac{*}{p(y), x = y \vdash p(y)}}{p(y), x = y \vdash p(x)}}{(x-y)^2 \leq 0, p(y), x = y \vdash p(x)}}{(x-y)^2 \leq 0, p(y) \vdash p(x)}}{(x-y)^2 \leq 0 \wedge p(y) \vdash p(x)}}{\vdash (x-y)^2 \leq 0 \wedge p(y) \rightarrow p(x)}$$

Outline

1. Learning Objectives
2. Sequent Calculus
   - Propositional Proof Rules
   - Soundness of Proof Rules
   - Proofs with Dynamics
   - Contextual Equivalence
   - Quantifier Proof Rules
   - A Sequent Proof for Single-hop Bouncing Balls
3. Real Arithmetic
   - Real Quantifier Elimination
   - Instantiating Real-Arithmetic Quantifiers
   - Weakening by Removing Assumptions
   - Abbreviating Terms to Reduce Complexity
   - Substituting Equations into Formulas
   - Creatively Cutting to Transform Questions
4. Summary

Summary: Proof Rules of Sequent Calculus

$$\neg R\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta} \qquad \wedge R\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\neg L\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta} \qquad \wedge L\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow R\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top R\ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

$$\rightarrow L\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \qquad \bot L\ \frac{}{\Gamma, \mathit{false} \vdash \Delta}$$

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(y \notin \Gamma,\Delta,\forall x\, p(x)) \qquad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}(\text{arbitrary term } e)$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}(\text{arbitrary term } e) \qquad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(y \notin \Gamma,\Delta,\exists x\, p(x))$$

Summary: Proof Rules of Sequent Calculus

$$\neg R\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta} \qquad \wedge R\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

$$\neg L\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta} \qquad \wedge L\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

$$\rightarrow R\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta} \qquad \text{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top R\ \frac{}{\Gamma \vdash \mathit{true}, \Delta}$$

$$\rightarrow L\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta} \qquad \text{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \qquad \bot L\ \frac{}{\Gamma, \mathit{false} \vdash \Delta}$$

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}(y \notin \Gamma,\Delta,\forall x\, p(x)) \qquad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}(\text{arbitrary term } e)$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}(\text{arbitrary term } e) \qquad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}(y \notin \Gamma,\Delta,\exists x\, p(x))$$

$$\mathbb{R}\ \frac{}{\Gamma \vdash \Delta} \qquad (\text{if } \bigwedge_{P \in \Gamma} P \rightarrow \bigvee_{Q \in \Delta} Q \text{ is valid in } \mathrm{FOL}_{\mathbb{R}})$$

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