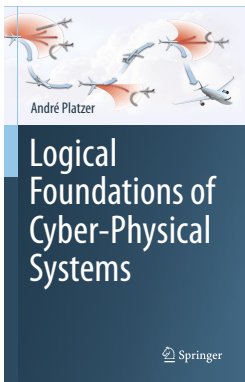


02: Differential Equations & Domains

Logical Foundations of Cyber-Physical Systems



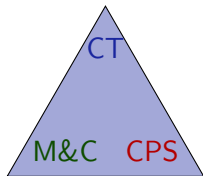
André Platzer



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- 2 Introduction
- 3 Differential Equations
- 4 Examples of Differential Equations
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 - Terms
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semantics of differential equations
descriptive power of differential equations
syntax versus semantics



continuous dynamics
differential equations
evolution domains
first-order logic

continuous operational effects

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Example (Vector field and one solution of a differential equation)

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

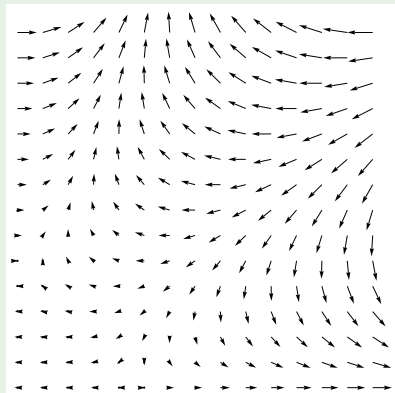
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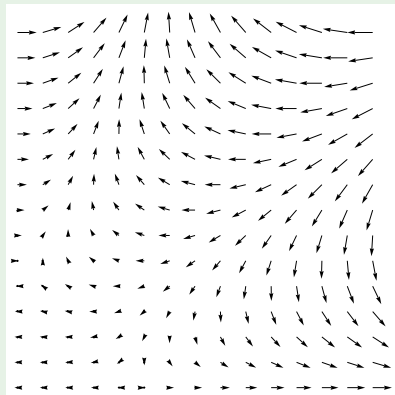


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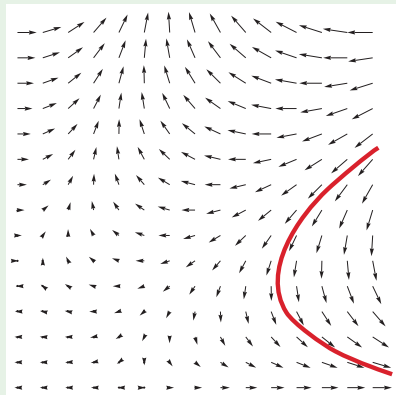


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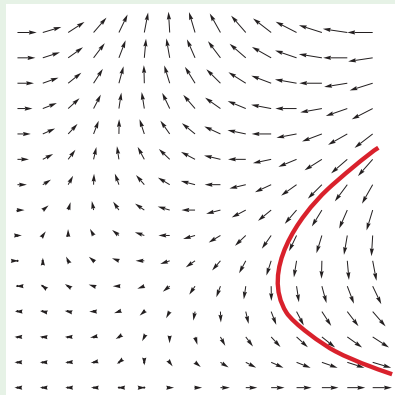


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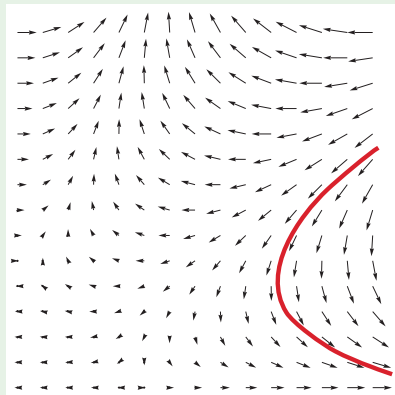


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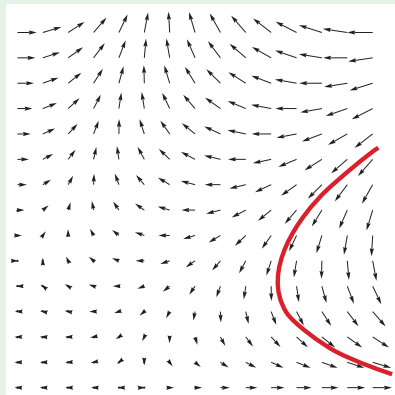
Your car's ODE: $x' = v, v' = a$

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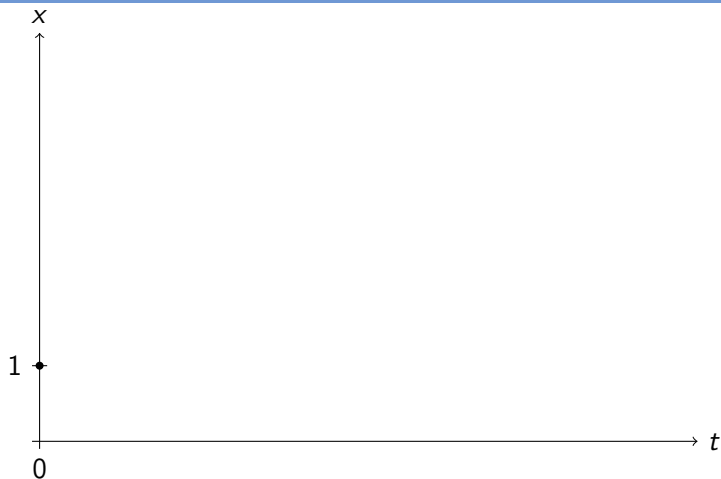
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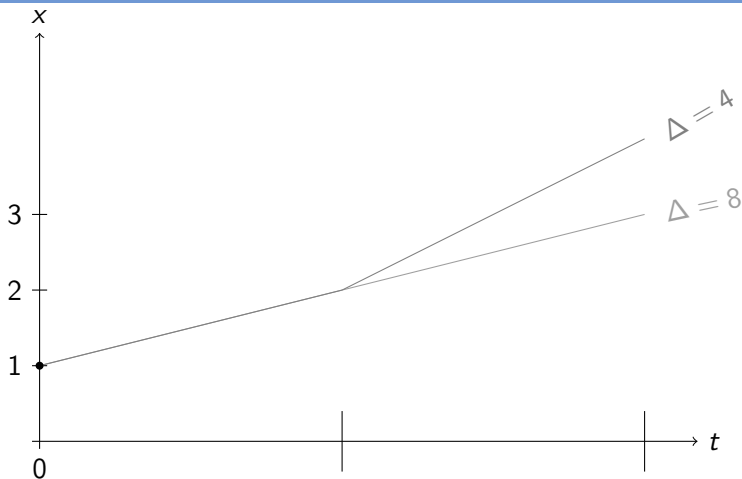
Well it's a wee bit more complicated



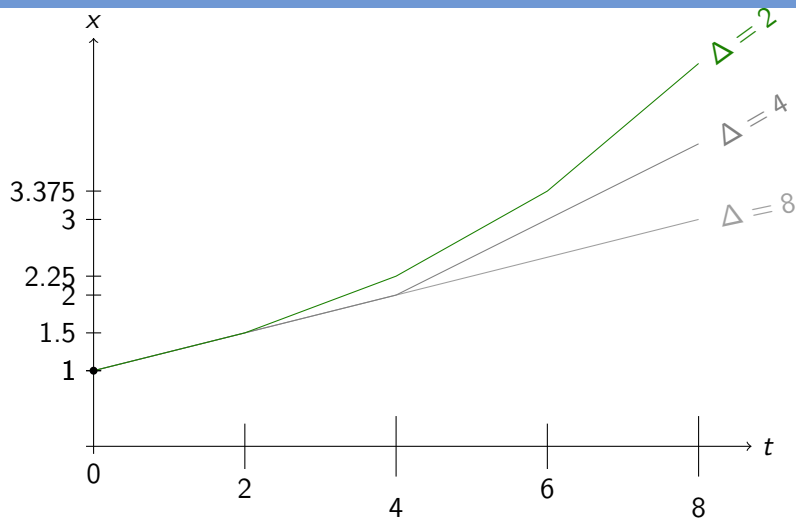
$$\begin{cases} x'(t) = \frac{1}{4}x(t) \\ x(0) = 1 \end{cases}$$



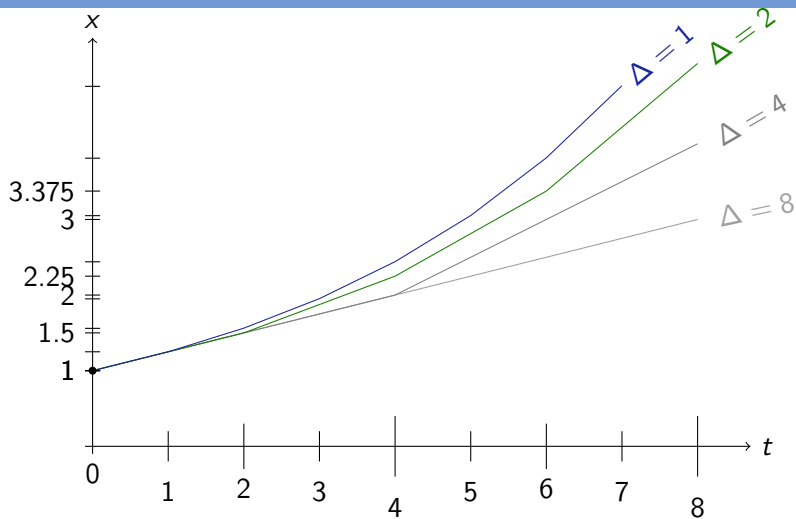
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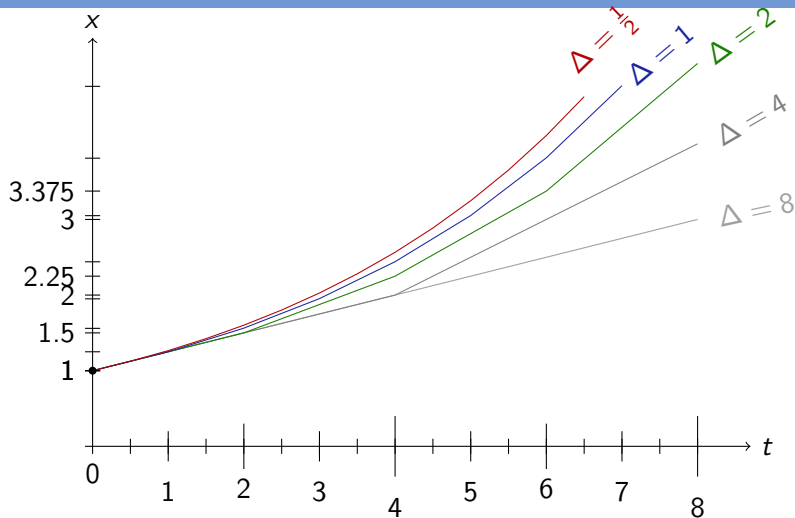
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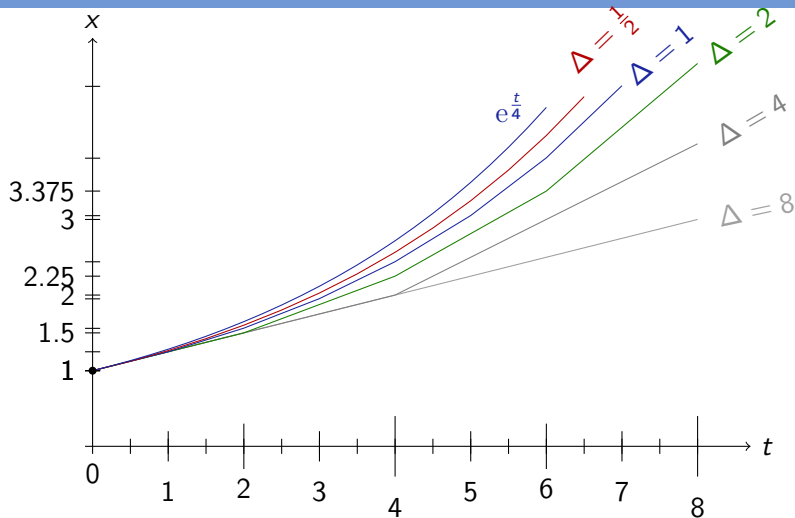
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- 1 What exactly is a vector field?
- 2 What does it mean to describe directions of evolution at *every* point in space?
- 3 Could these directions possibly contradict each other?

Importance of meaning

The physical impacts of CPSs do not leave much room for failure. We immediately want to get into the habit of studying the behavior and exact meaning of all relevant aspects of CPS.

Definition (Ordinary Differential Equation, ODE)

$f : D \rightarrow \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected set). Then $Y : I \rightarrow \mathbb{R}^n$ is *solution* of initial value problem (IVP)

$$\begin{pmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{pmatrix}$$

on the interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

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- 3 initial value $Y(t_0) = y_0$

If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.

If f continuous, then Y continuously differentiable.



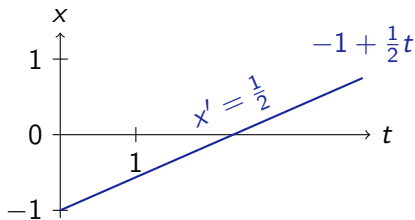
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Example (Initial value problem)

$$\begin{pmatrix} x'(t) = \frac{1}{2} \\ x(0) = -1 \end{pmatrix} \text{ has a solution}$$

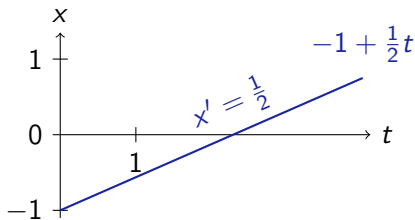
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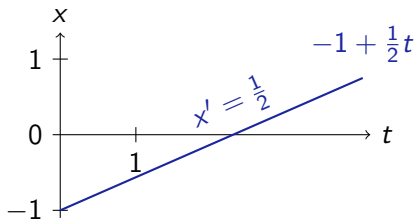
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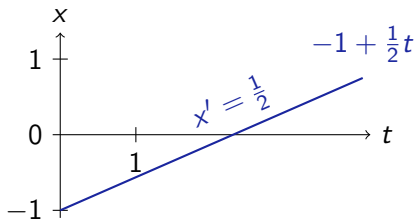
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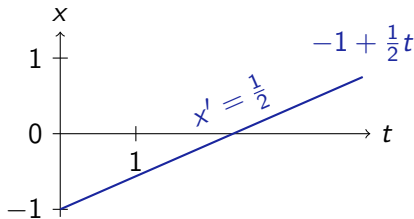
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Check by inserting solution into ODE+IVP.

$$\begin{cases} (x(t))' = (\frac{1}{2}t - 1)' = \frac{1}{2} \\ x(0) = \frac{1}{2} \cdot 0 - 1 = -1 \end{cases}$$

□





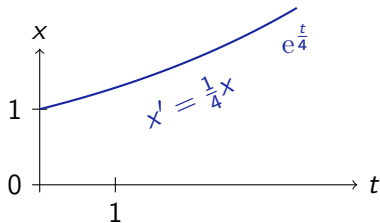
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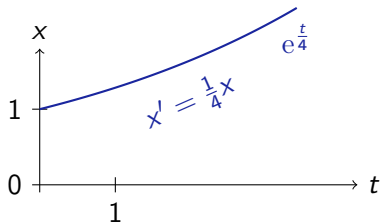
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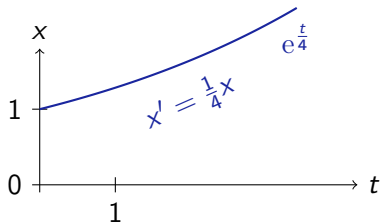
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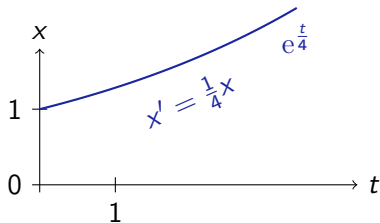
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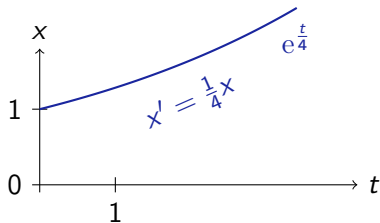


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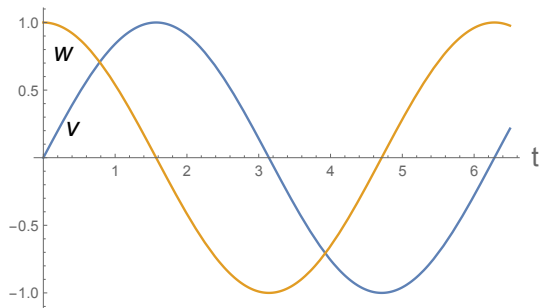
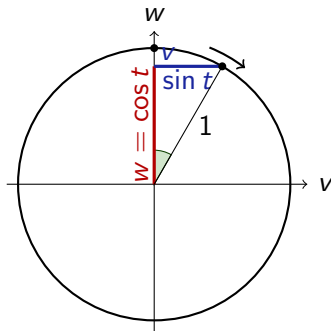


Example (Initial value problem)

$$\begin{pmatrix} v'(t) = w(t) \\ w'(t) = -v(t) \\ v(0) = 0 \\ w(0) = 1 \end{pmatrix} \text{ has solution}$$

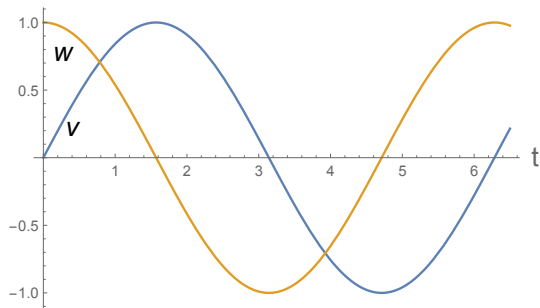
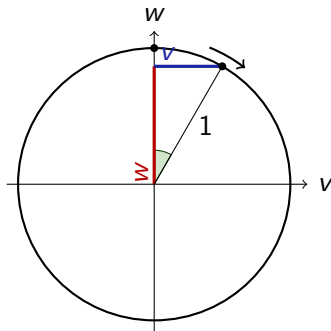
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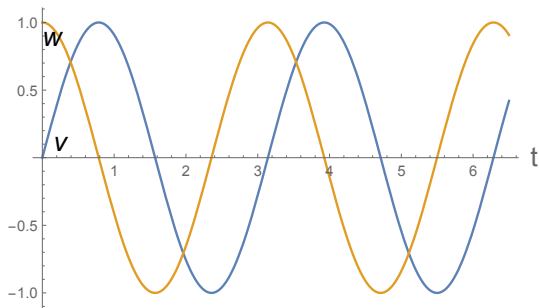
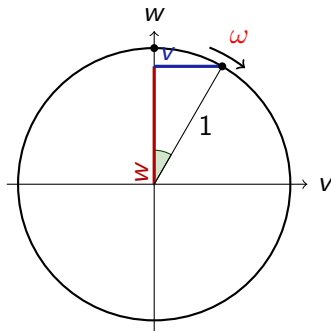
$$\begin{pmatrix} v'(t) = \omega w(t) \\ w'(t) = -\omega v(t) \\ v(0) = 0 \\ w(0) = 1 \end{pmatrix} \quad \text{has solution}$$



Example (Initial value problem)

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has solution $\begin{pmatrix} v(t) = \sin(\omega t) \\ w(t) = \cos(\omega t) \end{pmatrix}$

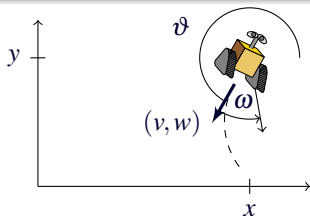


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ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$	$x(t) = \frac{x_0}{1 - tx_0}$
$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$
$y'(x) = -2xy, y(0) = 1$	$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$	$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$	$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$	$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$	$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$	$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$???
$x'(t) = e^{t^2}$	non-elementary



ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$	$x(t) = x_0 + 5t$
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$x' = x^2 + x^4$???
$x'(t) = e^{t^2}$	non-elementary

Descriptive power of differential equations

- 1 Solutions of differential equations can be much more involved than the differential equations themselves.
- 2 Representational and descriptive power of differential equations!
- 3 Simple differential equations can describe quite complicated physical processes.
- 4 Local description as the direction into which the system evolves.

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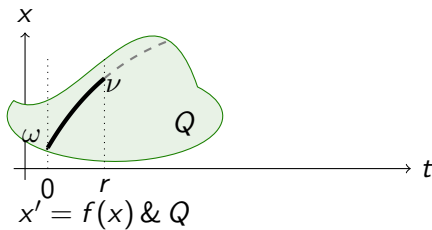
Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A differential equation $x' = f(x)$ with evolution domain Q is denoted by

$$x' = f(x) \& Q$$

conjunctive notation ($\&$) signifies that the system obeys the differential equation $x' = f(x)$ **and** the evolution domain Q .



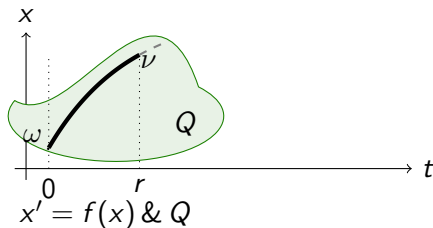
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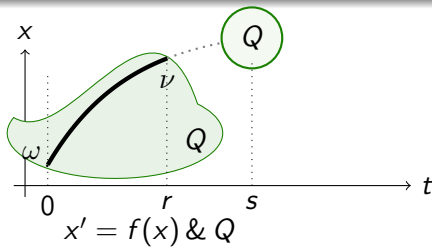
Enable Cyber to interact with Physics

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$$x' = v, v' = a, t' = 1 \& t \leq \varepsilon$$

stops at clock ε at the latest

$$x' = v, v' = a, t' = 1 \& v \geq 0$$

stops before velocity negative

$$x' = y, y' = x + y^2 \& \text{true}$$

no constraint

Define:
Terms

Cyber to interact with Physics

Define:
Formulas

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no constraint



Definition (Syntax of terms)

A *term* e is a polynomial term defined by the grammar:

$$e, \tilde{e} ::= x \mid c \mid e + \tilde{e} \mid e \cdot \tilde{e}$$

where e, \tilde{e} are terms, $x \in \mathcal{V}$ is a variable, $c \in \mathbb{Q}$ a rational number constant



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$$(\llbracket \cdot \rrbracket : \text{Trm} \rightarrow (\mathcal{S} \rightarrow \mathbb{R}))$$

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$$\omega \llbracket 4 + x \cdot 2 \rrbracket =$$

$$\text{if } \omega(x) = 5$$

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$$\omega \llbracket 4 + x \cdot 2 \rrbracket = \omega \llbracket 4 \rrbracket + \omega \llbracket x \rrbracket \cdot \omega \llbracket 2 \rrbracket = 4 + \omega(x) \cdot 2 = 14 \quad \text{if } \omega(x) = 5$$

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What about $x - y$?

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What about $x - y$? Defined as $x + (-1) \cdot y$

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What about x^4 ?

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What about x^4 ? Defined as $x \cdot x \cdot x \cdot x$

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What about x^n ?

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What about x^n ? Defined as $x \cdot x \cdot x \cdot x \cdot x \cdot \dots$, wait when do we stop???

Definition (Syntax of first-order logic formulas)

The *formulas* of *FOL of real arithmetic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid e = \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid P \leftrightarrow Q \mid \forall x P \mid \exists x P$$

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Greater-or-equal

Not

And

Or

Imply

Equiv

All

Exists

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Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω , written $\omega \models P$, defined inductively:

$$\omega \models e = \tilde{e} \quad \text{iff } \omega[e] = \omega[\tilde{e}]$$

$$\omega \models e \geq \tilde{e} \quad \text{iff } \omega[e] \geq \omega[\tilde{e}]$$

$$\omega \models \neg P \quad \text{iff } \omega \not\models P, \text{ i.e., if it is not the case that } \omega \models P$$

$$\omega \models P \wedge Q \quad \text{iff } \omega \models P \text{ and } \omega \models Q$$

$$\omega \models P \vee Q \quad \text{iff } \omega \models P \text{ or } \omega \models Q$$

$$\omega \models P \rightarrow Q \quad \text{iff } \omega \not\models P \text{ or } \omega \models Q$$

$$\omega \models \forall x P \quad \text{iff } \omega_x^d \models P \text{ for all } d \in \mathbb{R}$$

$$\omega \models \exists x P \quad \text{iff } \omega_x^d \models P \text{ for some } d \in \mathbb{R}$$

$$\omega_x^d(y) = \begin{cases} d & \text{if } y=x \\ \omega(y) & \text{if } y \neq x \end{cases}$$

$\omega \models P$ formula P is true in state ω

$\models P$ formula P is *valid*, i.e., true in all states ω , i.e., $\omega \models P$ for all ω

$\llbracket P \rrbracket = \{\omega : \omega \models P\}$ set of all states in which P is true

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$\omega \models P \wedge Q$ iff $\omega \models P$ and $\omega \models Q$

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$$\exists y (y^2 \leq x)$$

$$\text{for } \omega(x) = 5 \text{ and } \nu(x) = -5$$

Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω , written $\omega \models P$, defined inductively:

$$\omega \models e = \tilde{e} \quad \text{iff } \omega \llbracket e \rrbracket = \omega \llbracket \tilde{e} \rrbracket$$

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$\omega \models P$ formula P is true in state ω

$\models P$ formula P is *valid*, i.e., true in all states ω , i.e., $\omega \models P$ for all ω

$\llbracket P \rrbracket = \{\omega : \omega \models P\}$ set of all states in which P is true

$\omega \models \exists y (y^2 \leq x)$ but $\nu \not\models \exists y (y^2 \leq x)$ for $\omega(x) = 5$ and $\nu(x) = -5$

Definition (Semantics of first-order logic formulas)

First-order formula P is true in state ω , written $\omega \models P$, defined inductively:

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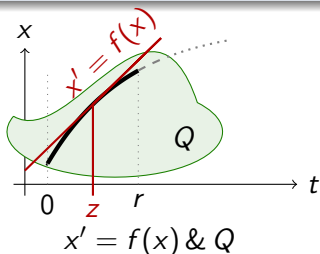
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Definition (Semantics of differential equations)

A function $\varphi : [0, r] \rightarrow \mathcal{S}$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \ \& \ Q$, written $\varphi \models x' = f(x) \ \& \ Q$, iff:

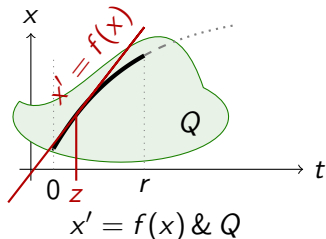
- 1 $\varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z)$ exists at all times $0 \leq z \leq r$
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- 3 $\varphi(z) = \varphi(0)$ except at x, x'



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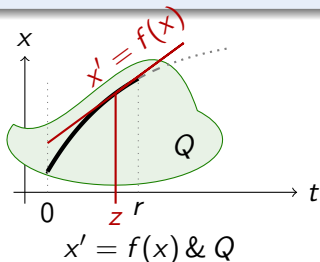
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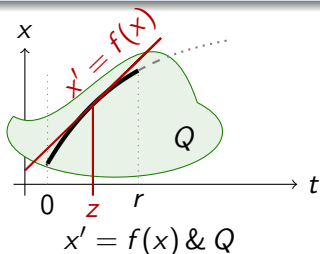
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- 1 Learning Objectives
- 2 Introduction
- 3 Differential Equations
- 4 Examples of Differential Equations
- 5 Domains of Differential Equations
 - Terms
 - First-Order Formulas
 - Continuous Programs
- 6 Summary

Definition (Syntax of terms)

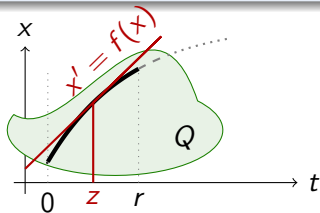
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Definition (Syntax of continuous programs)

A differential equation $x' = f(x)$ with evolution domain Q is denoted by

$$x' = f(x) \& Q$$




André Platzer.

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