13: Differential Invariants & Proof Theory

15-424: Foundations of Cyber-Physical Systems

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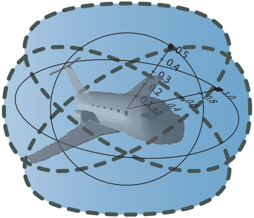

Outline

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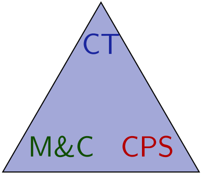
Learning Objectives

Differential Invariants & Proof Theory

limits of computation
proof theory for differential equations
provability of differential equations
nonprovability of differential equations
proofs about proofs
relativity theory of proofs
inform differential invariant search
intuition for differential equation proofs



none

core argumentative principles
tame analytic complexity

Outline

Differential Weakening

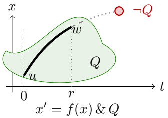
$$\frac{Q \vdash F}{P \vdash [x' = f(x) \,\&\, Q]F}$$

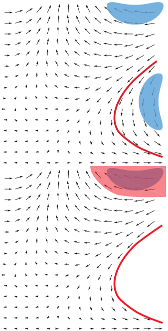
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$





DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \to F)$

DI $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [?Q]F\big) \leftarrow [x' = f(x) \,\&\, Q)](F)'$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

DE $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]F$

Outline

Relativity Theory of Proofs

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

But generalizations are helpful to find the right F in the first place:

$$\text{cut,MR}\ \frac{A \vdash F \quad F \vdash [x' = f(x) \,\&\, Q]F \quad F \vdash B}{A \vdash [x' = f(x) \,\&\, Q]B}$$

Compare Provability with Classes Ω of Differential Invariants

$\mathcal{DI}_\Omega$: properties provable with differential invariants in $\Omega \subseteq \{\geq, >, =, \wedge, \vee\}$

$\mathcal{A} \leq \mathcal{B}$ iff all properties provable with $\mathcal{A}$ are also provable somehow with $\mathcal{B}$

$\mathcal{A} \not\leq \mathcal{B}$ otherwise i.e. some property can be proved with $\mathcal{A}$ but not with $\mathcal{B}$

$\mathcal{A} \equiv \mathcal{B}$ iff $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$ so same deductive power

$\mathcal{A} < \mathcal{B}$ iff $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \not\leq \mathcal{A}$ so $\mathcal{A}$ has strictly less deductive power

Relativity Theory of Proofs

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$\mathcal{DI}_{e=k} \equiv \mathcal{DI}_{e=0}$ by considering $(e-k)=0$

But generalizations are helpful to find the right F in the first place:

$$\text{cut,MR}\frac{A \vdash F \quad F \vdash [x' = f(x) \,\&\, Q]F \quad F \vdash B}{A \vdash [x' = f(x) \,\&\, Q]B}$$

Compare Provability with Classes Ω of Differential Invariants

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Propositional Equivalences of Differential Invariants

Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is a propositional tautology then

$$F \text{ differential invariant of } x' = f(x) \,\&\, Q$$
$$\text{iff} \quad G \text{ differential invariant of } x' = f(x) \,\&\, Q$$

Proof.

□

Can use any propositional normal form

Propositional Equivalences of Differential Invariants

Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ *is a propositional tautology then*

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Proof.

$$\text{MR,cut}\;\frac{}{F \vdash [x' = f(x) \,\&\, Q]F}$$

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Proof.

$$\dfrac{\dfrac{}{G \vdash [x' = f(x) \,\&\, Q]G}\ \text{dI}}{F \vdash [x' = f(x) \,\&\, Q]F}\ \text{MR,cut}$$

□

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Proof.

$$\dfrac{\dfrac{\dfrac{}{Q \vdash [x':=f(x)](G)'}\;[' :=]}{G \vdash [x' = f(x) \,\&\, Q]G}\;\text{dI}}{F \vdash [x' = f(x) \,\&\, Q]F}\;\text{MR,cut}$$

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$F \leftrightarrow G$ propositionally equivalent, so $(F)' \leftrightarrow (G)'$ propositionally equivalent

□

Can use any propositional normal form

Propositional Equivalences of Differential Invariants

Lemma (Differential invariants and propositional logic)

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$$\dfrac{\dfrac{\dfrac{*}{Q \vdash [x':=f(x)](F)'}\;[':=]}{G \vdash [x' = f(x) \,\&\, Q]G}\;\text{dI}}{F \vdash [x' = f(x) \,\&\, Q]F}\;\text{MR,cut}$$

$F \leftrightarrow G$ propositionally equivalent, so
$(F)' \leftrightarrow (G)'$ propositionally equivalent
since $(F_1 \wedge F_2)' \equiv (F_1)' \wedge (F_2)' \ldots$

□

Can use any propositional normal form

Arithmetic Equivalences of Differential Invariants

Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is real-arithmetic equivalence then

$$
\begin{aligned}
& F \text{ differential invariant of } x' = f(x) \,\&\, Q \\
\text{iff} \quad & G \text{ differential invariant of } x' = f(x) \,\&\, Q
\end{aligned}
$$

Proof.

□

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Lemma (Differential invariants and propositional logic)

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Proof.

$$\text{dI}\ \frac{}{-5 \le x \land x \le 5 \vdash [x' = -x](-5 \le x \land x \le 5)}$$

□

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Proof.

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□

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Proof.

$$\dfrac{\dfrac{\overline{\vdash 0 \leq -x \land -x \leq 0}}{\vdash [x':=-x](0\leq x' \land x' \leq 0)}\,[':=]}{-5\leq x \land x \leq 5 \vdash [x' = -x](-5\leq x \land x \leq 5)}\,\text{dI}$$

□

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If $F \leftrightarrow G$ is *real-arithmetic* *equivalence then*

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Proof.

$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash 0 \leq -x \land -x \leq 0}}{\vdash [x' := -x](0 \leq x' \land x' \leq 0)}\,[':=]}{-5 \leq x \land x \leq 5 \vdash [x' = -x](-5 \leq x \land x \leq 5)}\,\text{dI}$$

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arithmetic equivalence $-5 \leq x \land x \leq 5 \leftrightarrow x^2 \leq 5^2$ □

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arithmetic equivalence $-5 \leq x \land x \leq 5 \leftrightarrow x^2 \leq 5^2$ □

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Lemma (Differential invariants and propositional logic)

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\qquad
\dfrac{\dfrac{\dfrac{}{\vdash -2x^2 \leq 0}\,\mathbb{R}}{\vdash [x' := -x]2xx' \leq 0}\,[':=]}{x^2 \leq 5^2 \vdash [x' = -x]x^2 \leq 5^2}\,\text{dI}$$

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Arithmetic Equivalences of Differential Invariants

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arithmetic equivalence $-5 \le x \land x \le 5 \leftrightarrow x^2 \le 5^2$ □

Arithmetic Equivalences of Differential Invariants

Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is real-arithmetic *equivalence then*

$$F \text{ differential invariant of } x' = f(x) \,\&\, Q$$
$$\text{iff } G \text{ differential invariant of } x' = f(x) \,\&\, Q$$

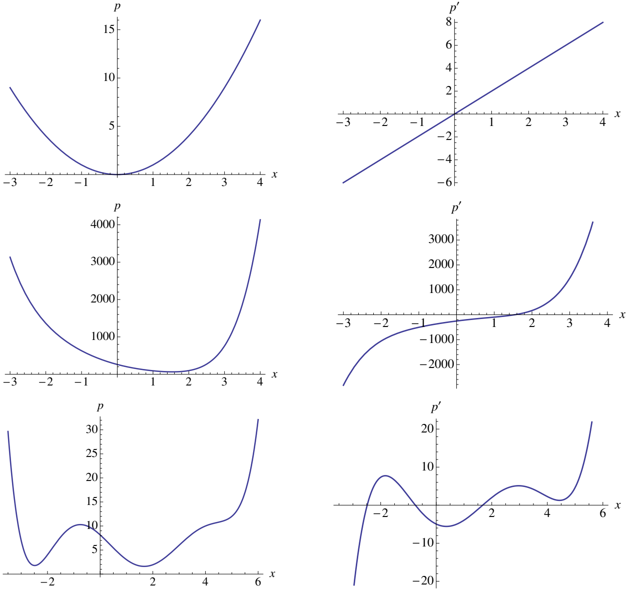
Proof.

$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash 0 \leq -x \land -x \leq 0}}{\vdash [x':=-x](0\leq x'\land x'\leq 0)}\,[':=]}{-5\leq x\land x\leq 5 \vdash [x' = -x](-5\leq x\land x\leq 5)}\,\text{dI}
\qquad
\dfrac{\dfrac{\dfrac{*}{\vdash -2x^2\leq 0}\,\mathbb{R}}{\vdash [x':=-x]2xx'\leq 0}\,[':=]}{x^2\leq 5^2 \vdash [x' = -x]x^2\leq 5^2}\,\text{dI}$$

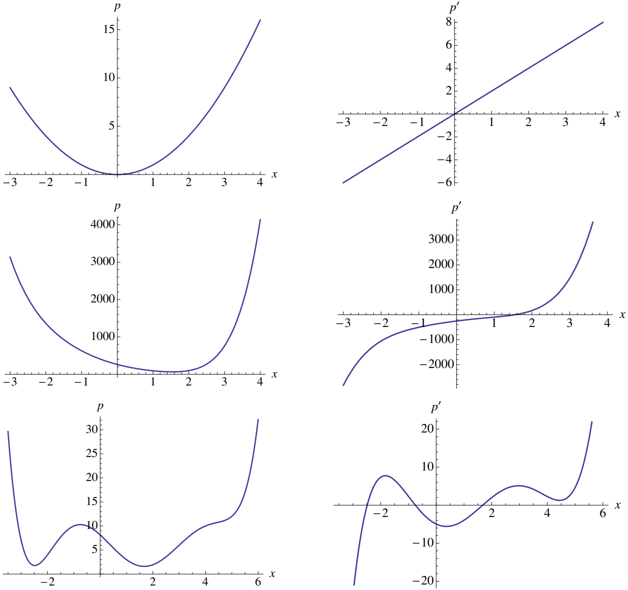
Despite arithmetic equivalence $-5\leq x\land x\leq 5 \leftrightarrow x^2\leq 5^2$ □

Differential structure matters! Higher degree helps here

Different Differential Structure for Equivalent Solutions ≥ 0

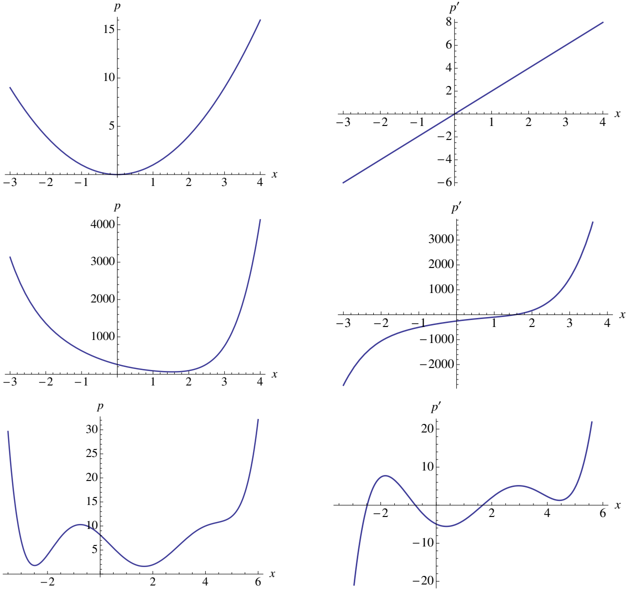


Different Differential Structure for Equivalent Solutions ≥ 0



Same $p \geq 0$.
But different $p' \geq 0$.

Different Differential Structure for Equivalent Solutions ≥ 0



Same $p \geq 0$.
But different $p' \geq 0$.

Can still normalize atomic formulas to $e = 0, e \geq 0, e > 0$

Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

$$\mathcal{DI}_{=} \qquad \mathcal{DI}_{=,\wedge,\vee}$$

Proof core.

-

-

Generalizations see [5, 1] □

Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

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-

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Differential Invariant Equations

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atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

- $e_1 = e_2 \vee k_1 = k_2$

- $e_1 = e_2 \wedge k_1 = k_2$

Generalizations see [5, 1] □

Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1] □

Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\land,\lor}$

Proof core.

- $e_1 = e_2 \lor k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x' := f(x)]((e_1)' = (e_2)' \land (k_1)' = (k_2)')$

- $e_1 = e_2 \land k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1] □

Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So$[x':=f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

  $\equiv[x':=f(x)]\big(((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0\big)$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1] □

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  $[x' := f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x' := f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

  $\equiv [x' := f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0)$
- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1] □

Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\land,\lor}$

Proof core.

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  $[x' := f(x)]((e_1)' = (e_2)' \land (k_1)' = (k_2)')$

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  $\equiv [x' := f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0)$

- $e_1 = e_2 \land k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

Generalizations see [5, 1] □

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- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

Generalizations see [5, 1] □

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atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $$[x':=f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$$

  $$\text{So}[x':=f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$$

  $$\equiv[x':=f(x)]\big(((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0\big)$$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

  $$[x':=f(x)]\big((e_1)' = (e_2)' \wedge (k_1)' = (k_2)'\big)$$

  $$\text{So}[x':=f(x)]\big(((e_1 - e_2)^2 + (k_1 - k_2)^2)'=0\big)$$

  $$\equiv\ [x':=f(x)]\big(2(e_1-e_2)((e_1)'-(e_2)') + 2(k_1-k_2)((k_1)'-(k_2)')=0\big)$$

Generalizations see [5, 1] □

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Proof core.

- $e_1 = e_2 \lor k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

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  $\equiv [x':=f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0)$

- $e_1 = e_2 \land k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

  $[x':=f(x)]((e_1)' = (e_2)' \land (k_1)' = (k_2)')$

  So$[x':=f(x)](((e_1 - e_2)^2 + (k_1 - k_2)^2)'=0)$

  $\equiv [x':=f(x)](2(e_1 - e_2)((e_1)' - (e_2)') + 2(k_1 - k_2)((k_1)' - (k_2)')=0)$

Generalizations see [5, 1] □

Differential Invariant Equations

Proposition (Equational deductive power [5, 1])

atomic equations are enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee}$

Proof core.

- $e_1 = e_2 \vee k_1 = k_2 \leftrightarrow (e_1 - e_2)(k_1 - k_2) = 0$

  $[x' := f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x' := f(x)]((e_1 - e_2)(k_1 - k_2))' = 0$

  $\equiv [x' := f(x)](((e_1)' - (e_2)')(k_1 - k_2) + (e_1 - e_2)((k_1)' - (k_2)') = 0)$

- $e_1 = e_2 \wedge k_1 = k_2 \leftrightarrow (e_1 - e_2)^2 + (k_1 - k_2)^2 = 0$

  $[x' := f(x)]((e_1)' = (e_2)' \wedge (k_1)' = (k_2)')$

  So $[x' := f(x)](((e_1 - e_2)^2 + (k_1 - k_2)^2)' = 0)$

  $\equiv [x' := f(x)](2(e_1 - e_2)((e_1)' - (e_2)') + 2(k_1 - k_2)((k_1)' - (k_2)') = 0)$

Generalizations see [5, 1] □

Equational

Proposition (Equational [1])

$$\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} \quad \mathcal{DI} \quad \mathcal{DI}_{\geq} \quad \mathcal{DI}_{=}$$

Proof core.

□

Equational Incompleteness

Proposition (Equational incompleteness [1])

Equations are not enough: $\mathcal{DI}_= \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_\geq \not\leq \mathcal{DI}_=$

Proof core.

$\square$

Equational Incompleteness

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Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

Provable with $\mathcal{DI}_{\geq}$ Unprovable with $\mathcal{DI}_{=}$

□

Equational Incompleteness

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Proof core.

Provable with $\mathcal{DI}_\geq$ | Unprovable with $\mathcal{DI}_=$

$$\text{dI}\ \frac{}{x \geq 0 \vdash [x' = 5]x \geq 0}$$

□

Equational Incompleteness

Proposition (Equational incompleteness [1])

Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

Provable with $\mathcal{DI}_{\geq}$ Unprovable with $\mathcal{DI}_{=}$

$$\dfrac{[{}':=]\ \dfrac{}{\vdash [x':=5]x' \geq 0}}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{dI}$$

□

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Proof core.

Provable with $\mathcal{DI}_{\geq}$ Unprovable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{\dfrac{}{\vdash 5 \geq 0}\;\mathbb{R}}{\vdash [x':=5]x' \geq 0}\;[':=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\;\text{dI}$$

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Proof core.

Provable with $\mathcal{DI}_\geq$

$$\text{dI}\ \dfrac{[':=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash 5 \geq 0}}{\vdash [x':=5]x' \geq 0}}{x \geq 0 \vdash [x' = 5]x \geq 0}$$

Unprovable with $\mathcal{DI}_=$

□

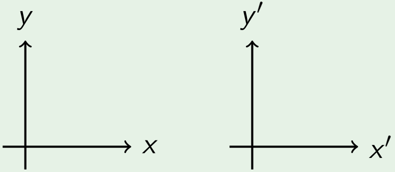
Proving Differences in Set Theory & Linear Algebra

Example (Sets Bijective or Not)

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$$

$$\mathsf{a} \longrightarrow \mathsf{b} \longrightarrow \mathsf{c} \longrightarrow \mathsf{d} \longrightarrow \mathsf{e} \longrightarrow \mathsf{f}$$

Example (Vector Spaces Isomorphic or Not)



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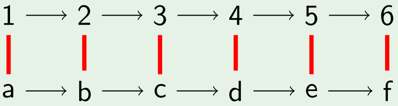

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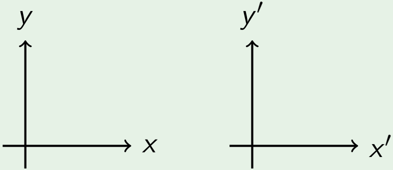

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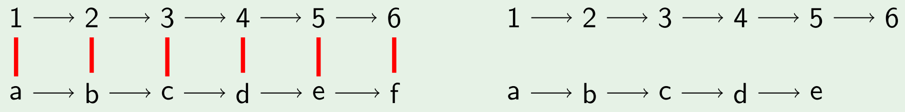

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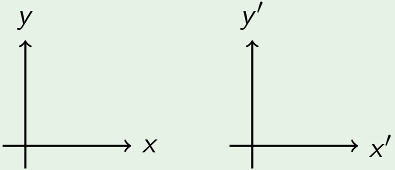

Proving Differences in Set Theory & Linear Algebra

Example (Sets Bijective or Not)

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$a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow e \longrightarrow f$

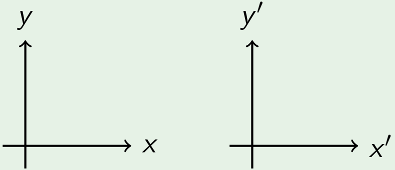
$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$

$a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow e$

criterion: cardinality $|\{1, \ldots, 6\}| = 6 \neq |\{a, b, c, d, e\}| = 5$

Need an indirect criterion especially if these sets are infinite

Example (Vector Spaces Isomorphic or Not)



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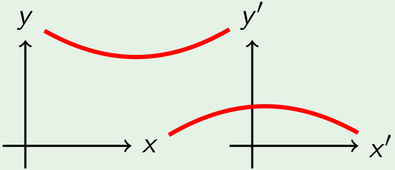

Proving Differences in Set Theory & Linear Algebra

Example (Sets Bijective or Not)

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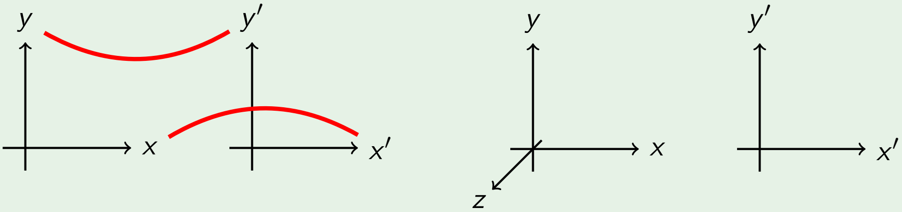
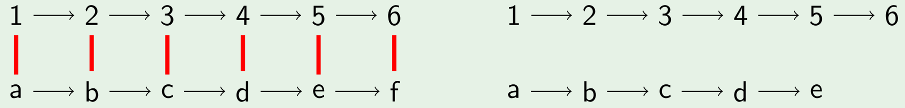


Proving Differences in Set Theory & Linear Algebra

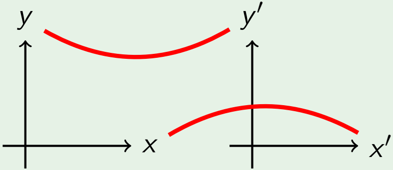
Example (Sets Bijective or Not)

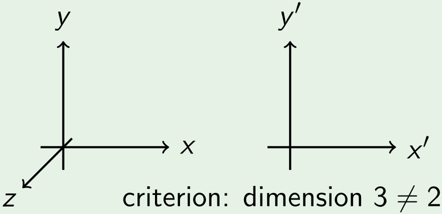


criterion: cardinality $|\{1,\dots,6\}| = 6 \neq |\{a,b,c,d,e\}| = 5$

Need an indirect criterion especially if these sets are infinite

Example (Vector Spaces Isomorphic or Not)





criterion: dimension $3 \neq 2$

Equational Incompleteness

Proposition (Equational incompleteness [1])

Equations are not enough: $\mathcal{DI}_{=} \equiv \mathcal{DI}_{=,\wedge,\vee} < \mathcal{DI}$ *because* $\mathcal{DI}_{\geq} \not\leq \mathcal{DI}_{=}$

Proof core.

Provable with $\mathcal{DI}_{\geq}$ Unprovable with $\mathcal{DI}_{=}$

$$\text{dI}\ \dfrac{[':=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash 5 \geq 0}}{\vdash [x':=5]x' \geq 0}}{x \geq 0 \vdash [x' = 5]x \geq 0}$$

□

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$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x' := 5] x' \geq 0}\ [':=]}{x \geq 0 \vdash [x' = 5] x \geq 0}\ \text{dI}$$

$$\dfrac{}{x \geq 0 \vdash [x' = 5] x \geq 0}\ \text{cut,MR}$$

□

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$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0}\;\mathbb{R}}{\vdash [x':=5]x' \geq 0}\;[':=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\;\text{dI}$$

Unprovable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{}{p(x) = 0 \vdash [x' = 5]p(x) = 0}\;\text{dI}}{x \geq 0 \vdash [x' = 5]x \geq 0}\;\text{cut,MR}$$

□

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Unprovable with $\mathcal{DI}_=$

$$\dfrac{\dfrac{\overline{\vdash [x':=5](p(x))' = 0}}{p(x) = 0 \vdash [x' = 5]p(x) = 0}\ \text{dI}}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{cut,MR}$$

□

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Unprovable with $\mathcal{DI}_{=}$

$$\text{cut,MR}\ \dfrac{\text{dI}\ \dfrac{\dfrac{???}{\vdash [x':=5](p(x))' = 0}}{p(x) = 0 \vdash [x' = 5]p(x) = 0}}{x \geq 0 \vdash [x' = 5]x \geq 0}$$

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Unprovable with $\mathcal{DI}_=$

$$\dfrac{\dfrac{\dfrac{???}{\vdash [x':=5](p(x))' = 0}}{p(x) = 0 \vdash [x' = 5]p(x) = 0}\ \text{dI}}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{cut,MR}$$

Univariate polynomial $p(x)$ is 0 if 0 on all $x \geq 0$ □

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Proof core.

Provable with $\mathcal{DI}_{\geq}$

$$\cfrac{\cfrac{\cfrac{*}{\vdash 5 \geq 0}\ \mathbb{R}}{\vdash [x':=5]x' \geq 0}\ [':=]}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{dI}$$

Unprovable with $\mathcal{DI}_{=}$

$$\cfrac{\cfrac{\cfrac{???}{\vdash [x':=5](p(x))' = 0}\ \text{dI}}{p(x) = 0 \vdash [x' = 5]p(x) = 0}}{x \geq 0 \vdash [x' = 5]x \geq 0}\ \text{cut,MR}$$

Univariate polynomial $p(x)$ is 0 if 0 on all $x \geq 0$

Likewise for indirect proofs [1]. □

Strict Inequality

Proposition (Strict barrier)

$\mathcal{DI}_>$ $\mathcal{DI}$ $\mathcal{DI}_=$ $\mathcal{DI}_>$

Proof core.

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

Provable with $\mathcal{DI}_{=}$

Unprovable with $\mathcal{DI}_{>}$

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

Provable with $\mathcal{DI}_{=}$ Unprovable with $\mathcal{DI}_{>}$

$$\text{dI}\ \frac{}{v^2{+}w^2{=}c^2 \vdash [v' = w, w' = -v]v^2{+}w^2{=}c^2}$$

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

Provable with $\mathcal{DI}_{=}$

Unprovable with $\mathcal{DI}_{>}$

$$\dfrac{[{}':=]\dfrac{}{\vdash [v':=w][w':=-v]2vv'+2ww'=0}}{v^2+w^2=c^2 \vdash [v'=w, w'=-v]v^2+w^2=c^2}\ \text{dI}$$

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

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Proof core.

Provable with $\mathcal{DI}_{=}$

Unprovable with $\mathcal{DI}_{>}$

$$\dfrac{\dfrac{\dfrac{}{\vdash 2vw + 2w(-v) = 0}\;\mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' = 0}\;[':=]}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\;\text{dI}$$

□

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Proof core.

Provable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 2vw + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' = 0}\ [':=]}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\ \text{dI}$$

Unprovable with $\mathcal{DI}_{>}$

□

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_> < \mathcal{DI}$ *because* $\mathcal{DI}_= \not\leq \mathcal{DI}_>$

Proof core.

Provable with $\mathcal{DI}_=$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 2vw + 2w(-v) = 0}\,\mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' = 0}\,[':=]}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\,\text{dI}$$

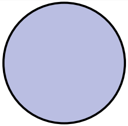
Unprovable with $\mathcal{DI}_>$

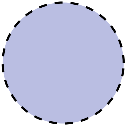
$e > 0$ is open set.

$v^2+w^2=c^2$ is a closed set

□

closed $v^2+w^2\leq 1$
with boundary





open $v^2+w^2<1$
without boundary

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

Provable with $\mathcal{DI}_{=}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 2vw + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v'{:=}w][w'{:=}-v]2vv' + 2ww' = 0}\ [':=]}{v^2{+}w^2{=}c^2 \vdash [v' = w, w' = -v]v^2{+}w^2{=}c^2}\ \mathrm{dI}$$

$v^2{+}w^2{=}c^2$ is a closed set

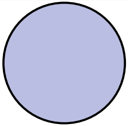
Unprovable with $\mathcal{DI}_{>}$

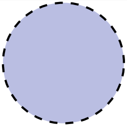
$e > 0$ is open set.

Only *true* and *false* are both

□

closed $v^2{+}w^2{\leq}1$ with boundary





open $v^2{+}w^2{<}1$ without boundary

Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

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Proof core.

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Unprovable with $\mathcal{DI}_{>}$

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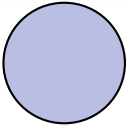
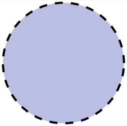
Only *true* and *false* are both

but don't help proof.

□

closed $v^2+w^2 \leq 1$
with boundary





open $v^2+w^2 < 1$
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Strict Inequality Incompleteness

Proposition (Strict barrier incompleteness)

Strict inequalities are not enough: $\mathcal{DI}_{>} < \mathcal{DI}$ *because* $\mathcal{DI}_{=} \not\leq \mathcal{DI}_{>}$

Proof core.

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$$\dfrac{\dfrac{\dfrac{*}{\vdash 2vw + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v' := w][w' := -v]2vv' + 2ww' = 0}\ [':=]}{v^2+w^2=c^2 \vdash [v' = w, w' = -v]v^2+w^2=c^2}\ \text{dI}$$

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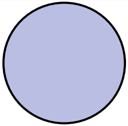
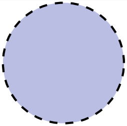
Likewise for indirect proofs [1]. □

closed $v^2+w^2 \leq 1$ with boundary





open $v^2+w^2<1$ without boundary

Differential Invariant Equations to Inequalities

Proposition (Equational)

$\mathcal{DI}_{=,\wedge,\vee}$ $\mathcal{DI}_{\geq}$

Proof core.

□

Differential Invariant Equations to Inequalities

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

□

Differential Invariant Equations to Inequalities

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Provable with $\mathcal{DI}_{\geq}$

□

Differential Invariant Equations to Inequalities

Proposition (Equational definability)

Equations are definable by weak inequalities: $\mathcal{DI}_{=,\wedge,\vee} \leq \mathcal{DI}_{\geq}$

Proof core.

Provable with $\mathcal{DI}_{=}$

$$\text{dI}\;\frac{}{e = 0 \vdash [x' = f(x)\,\&\,Q]e = 0}$$

Provable with $\mathcal{DI}_{\geq}$

□

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Provable with $\mathcal{DI}_{\geq}$

$$\text{dI}\dfrac{}{-e^2 \geq 0 \vdash [x' = f(x)\,\&\,Q](-e^2 \geq 0)}$$

$\square$

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$$\text{dI}\dfrac{\dfrac{*}{Q \vdash [x' := f(x)](e)' = 0}}{e = 0 \vdash [x' = f(x) \,\&\, Q]e = 0}$$

Provable with $\mathcal{DI}_{\geq}$

$$\text{dI}\dfrac{\dfrac{}{Q \vdash [x' := f(x)] - 2e(e)' \geq 0}}{-e^2 \geq 0 \vdash [x' = f(x) \,\&\, Q](-e^2 \geq 0)}$$

□

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Likewise for indirect proofs [1]. □

Local view of logic on differentials is crucial for this proof.
Degree increases

Differential Invariant Atoms

Theorem (Atomic)

$\mathcal{DI}_{\geq} \quad \mathcal{DI}_{\geq,\wedge,\vee}$ and $\mathcal{DI}_{>} \quad \mathcal{DI}_{>,\wedge,\vee}$

Proof idea.

□

Differential Invariant Atoms

Theorem (Atomic incompleteness)

Atomic inequalities not enough: $\mathcal{DI}_{\geq} < \mathcal{DI}_{\geq,\wedge,\vee}$ *and* $\mathcal{DI}_{>} < \mathcal{DI}_{>,\wedge,\vee}$

Proof idea.

□

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Unprovable with $\mathcal{DI}_{\geq}$

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Proof idea.

Provable with $\mathcal{DI}_{\geq,\wedge,\vee}$

Unprovable with $\mathcal{DI}_{\geq}$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5 \geq 0 \wedge y^2 \geq 0}\ \mathbb{R}}{\vdash [x':=5][y':=y^2](x'\geq 0 \wedge y' \geq 0)}\ [':=]}{x\geq 0\wedge y\geq 0 \vdash [x' = 5, y' = y^2](x\geq 0\wedge y\geq 0)}\ \text{dI}$$

□

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Unprovable with $\mathcal{DI}_{\geq}$
$p(x,y)\geq 0 \leftrightarrow x\geq 0 \wedge y \geq 0$
impossible since this implies
$p(x,0)\geq 0 \leftrightarrow x \geq 0$
so $p(x,0)$ is 0

□

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Proof idea.

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Substantial remaining parts of the proof shown elsewhere [1]. □

Differential Invariant Atoms

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Substantial remaining parts of the proof shown elsewhere [1]. □

dC still possible here but more involved argument separates.

Outline

Deductive Power of Differential Cuts & Differential Ghosts

Theorem (Gentzen's Cut Elimination) (1935)

$$\frac{A \vdash B \lor C \quad A \land C \vdash B}{A \vdash B} \qquad \textit{cut can be eliminated}$$

Theorem (No Differential Cut Elimination) (LMCS 2012)

Deductive power with differential cuts exceeds deductive power without.

$$\mathcal{DI} + DC > \mathcal{DI}$$

Theorem (Auxiliary Differential Variables) (LMCS 2012)

Deductive power with differential ghosts exceeds power without.

$$\mathcal{DI} + DC + DG > \mathcal{DI} + DC$$

LMCS'12

$$\text{dI}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

Ex: The Need for Differential Cuts

$$\dfrac{[{}':=]\ \dfrac{}{\vdash [x':=(x-2)^4+y^5][y':=y^2]2x^2\textcolor{red}{x'} \geq 0}}{\text{dI}\quad x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]x^3 \geq -1}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash 2x^2((x-2)^4+y^5)\geq 0}}{\vdash [x':=(x-2)^4+y^5][y':=y^2]2x^2x'\geq 0}\;[':=]}{x^3\geq -1\land y^5\geq 0\vdash [x'=(x-2)^4+y^5, y'=y^2]x^3\geq -1}\;\text{dI}$$

Ex: The Need for Differential Cuts

$$\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash 2x^2((x-2)^4+y^5)\geq 0}}{\vdash [x' := (x-2)^4+y^5][y' := y^2]2x^2x' \geq 0}\;[':=]}{x^3\geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4+y^5, y' = y^2]x^3 \geq -1}\;\text{dI}$$

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Have to know something about y^5

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

Ex: Differential Cuts

$$\text{dC}\ \dfrac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{[':=]\ \dfrac{}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 {\color{red}y'} \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]{\color{red}y^5 \geq 0}}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{[':=]\ \dfrac{\mathbb{R}\ \dfrac{}{\vdash 5y^4 {\color{red}y^2} \geq 0}}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 {\color{red}y'} \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]{\color{red}y^5 \geq 0}}$$

$$\text{dC}\ \frac{}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]x^3 \geq -1}$$

$$\text{dI}\ \dfrac{\text{[}'{:=}\text{]}\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash 5y^4 y^2 \geq 0}}{\vdash [x':=(x-2)^4 + y^5][y':=y^2]5y^4 y' \geq 0}}{y^5 \geq 0 \vdash [x' = (x-2)^4 + y^5, y' = y^2]y^5 \geq 0}$$

Ex: Differential Cuts

$$\text{dI}\ \dfrac{}{x^3 \geq -1 \vdash [x' = (x-2)^4 + y^5, y' = y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \ \triangleright}$$

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Ex: Differential Cuts

$$\dfrac{\dfrac{\dfrac{}{y^5 \geq 0 \vdash [x':=(x-2)^4+y^5][y':=y^2]2x^2x' \geq 0}[':=]}{x^3 \geq -1 \vdash [x'=(x-2)^4+y^5, y'=y^2 \,\&\, y^5 \geq 0]x^3 \geq -1 \;\triangleright}\text{dI}}{x^3 \geq -1 \land y^5 \geq 0 \vdash [x'=(x-2)^4+y^5, y'=y^2]x^3 \geq -1}\text{dC}$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash 5y^4y^2 \geq 0}\mathbb{R}}{\vdash [x':=(x-2)^4+y^5][y':=y^2]5y^4y' \geq 0}[':=]}{y^5 \geq 0 \vdash [x'=(x-2)^4+y^5, y'=y^2]y^5 \geq 0}\text{dI}$$

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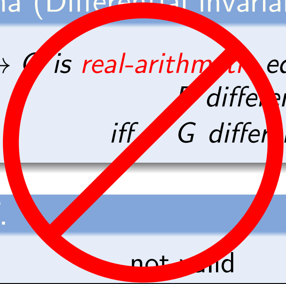
Outline

Arithmetic Equivalences of Differential Invariants



Lemma (Differential invariants and propositional logic)

If $F \leftrightarrow G$ is real-arithmetic *equivalence then*

$$F \text{ differential invariant of } x' = f(x) \,\&\, Q$$
$$\text{iff } G \text{ differential invariant of } x' = f(x) \,\&\, Q$$

Proof.

$$\text{dI}\ \dfrac{[':=]\ \dfrac{\dfrac{\text{not valid}}{\vdash 0 \leq -x \land -x \leq 0}}{\vdash [x' := -x](0 \leq x' \land x' \leq 0)}}{-5 \leq x \land x \leq 5 \vdash [x' = -x](-5 \leq x \land x \leq 5)} \qquad \text{dI}\ \dfrac{[':=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash -2x^2 \leq 0}}{\vdash [x' := -x]2xx' \leq 0}}{x^2 \leq 5^2 \vdash [x' = -x]x^2 \leq 5^2}$$

Despite arithmetic equivalence $-5 \leq x \land x \leq 5 \leftrightarrow x^2 \leq 5^2$ □

Differential structure matters! Higher degree helps here

Curves Playing with Norms and Degrees

$$\text{dC}\ \frac{}{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1]\|(x, y)\|_\infty \leq t}$$

$$A \stackrel{\text{def}}{\equiv} v^2+w^2 \leq 1 \land x=y=t=0$$

$$\|(x, y)\|_\infty \leq t \stackrel{\text{def}}{\equiv} -t \leq x \leq t \land -t \leq y \leq t$$ Supremum norm

$$\|(x, y)\|_2 \leq t \stackrel{\text{def}}{\equiv} x^2 + y^2 \leq t^2$$ Euclidean norm

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$$\text{dI} \quad \triangleleft \quad A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1 \,\&\, v^2+w^2\leq 1]\|(x,y)\|_\infty \leq t$$
$$\text{dC} \quad \overline{A \vdash [x' = v, y' = w, v' = \omega w, w' = -\omega v, t' = 1]\|(x,y)\|_\infty \leq t}$$

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Curves Playing with Norms and Degrees

$$\dfrac{\dfrac{\overline{v^2+w^2\leq 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t'\leq x'\leq t' \land -t'\leq y'\leq t')}\;[':=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\leq 1]\|(x,y)\|_\infty \leq t}\;\text{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_\infty \leq t}\;\text{dC}$$

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Curves Playing with Norms and Degrees

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Curves Playing with Norms and Degrees

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$$\dfrac{\dfrac{}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\le 1]\|(x,y)\|_2 \le t}\;\mathrm{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_2 \le t}\;\mathrm{dC}$$

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Curves Playing with Norms and Degrees

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{v^2+w^2\le1 \vdash -1\le v\le 1 \land -1\le w\le 1}\ \mathbb{R}}{v^2+w^2\le1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t'\le x'\le t' \land -t'\le y'\le t')}\ [':=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\le1]\|(x,y)\|_\infty \le t}\ \text{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_\infty \le t}\ \text{dC}$$

$$\dfrac{\dfrac{\overline{v^2+w^2\le1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](2xx'+2yy'\le 2tt')}\ [':=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\le1]\|(x,y)\|_2 \le t}\ \text{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_2 \le t}\ \text{dC}$$

$$A \stackrel{\text{def}}{\equiv} v^2+w^2\le1 \land x=y=t=0$$

$$\|(x,y)\|_\infty \le t \stackrel{\text{def}}{\equiv} -t\le x\le t \land -t\le y\le t \qquad \text{Supremum norm}$$

$$\|(x,y)\|_2 \le t \stackrel{\text{def}}{\equiv} x^2+y^2\le t^2 \qquad \text{Euclidean norm}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{v^2+w^2\leq 1 \vdash -1 \leq v \leq 1 \land -1 \leq w \leq 1}\;\mathbb{R}}{v^2+w^2\leq 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](-t'\leq x'\leq t' \land -t'\leq y'\leq t')}\;[':=]}{\triangleleft \qquad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\leq 1]\|(x,y)\|_\infty \leq t}\;\mathrm{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_\infty \leq t}\;\mathrm{dC}$$

$$\dfrac{\dfrac{\dfrac{\overline{v^2+w^2\leq 1 \vdash 2xv+2yw \leq 2t}}{v^2+w^2\leq 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](2xx'+2yy'\leq 2tt')}\;[':=]}{\triangleleft \qquad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\leq 1]\|(x,y)\|_2 \leq t}\;\mathrm{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_2 \leq t}\;\mathrm{dC}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\text{not valid}}{v^2+w^2\le 1 \vdash 2xv+2yw\le 2t}}{v^2+w^2\le 1 \vdash [x':=v][y':=w][v':=\omega w][w':=-\omega v][t':=1](2xx'+2yy'\le 2tt')}\ [':=]}{\triangleleft \quad A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1 \,\&\, v^2+w^2\le 1]\|(x,y)\|_2 \le t}\ \text{dI}}{A \vdash [x'=v, y'=w, v'=\omega w, w'=-\omega v, t'=1]\|(x,y)\|_2 \le t}\ \text{dC}$$

$$A \stackrel{\text{def}}{\equiv} v^2+w^2\le 1 \land x=y=t=0$$

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$\|(x,y)\|_2 \le t \stackrel{\text{def}}{\equiv} x^2+y^2\le t^2$ — Euclidean norm

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Lower degree helps here

Interreducing Norms in Dimension n

$$\forall x\,\forall y\,(\|(x,y)\|_\infty \le \|(x,y)\|_2 \le \sqrt{n}\|(x,y)\|_\infty)$$

$$\forall x\,\forall y\,(\frac{1}{\sqrt{n}}\|(x,y)\|_2 \le \|(x,y)\|_\infty \le \|(x,y)\|_2)$$

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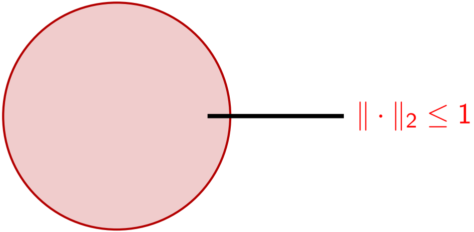

Interreducing Norms in Dimension n

$$\forall x\, \forall y\, (\|(x,y)\|_\infty \leq \|(x,y)\|_2 \leq \sqrt{n}\|(x,y)\|_\infty)$$

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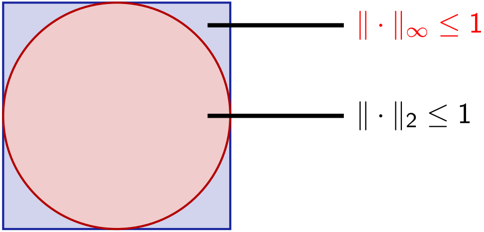

Interreducing Norms in Dimension n

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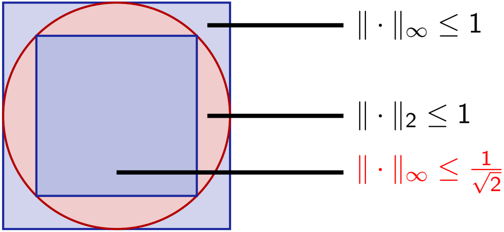

Interreducing Norms in Dimension n

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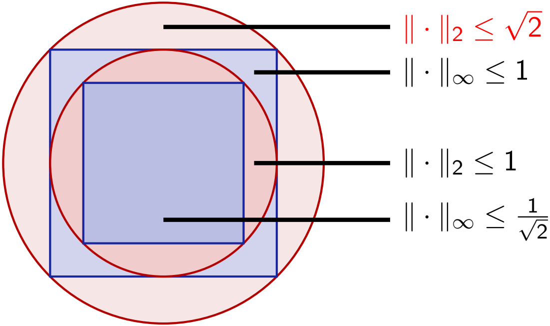

Interreducing Norms in Dimension n

$$\forall x\, \forall y\, (\|(x,y)\|_\infty \leq \|(x,y)\|_2 \leq \sqrt{n}\|(x,y)\|_\infty)$$

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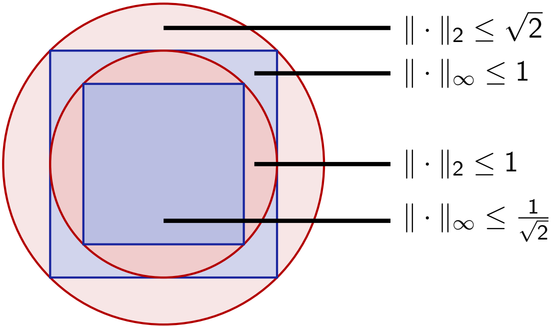
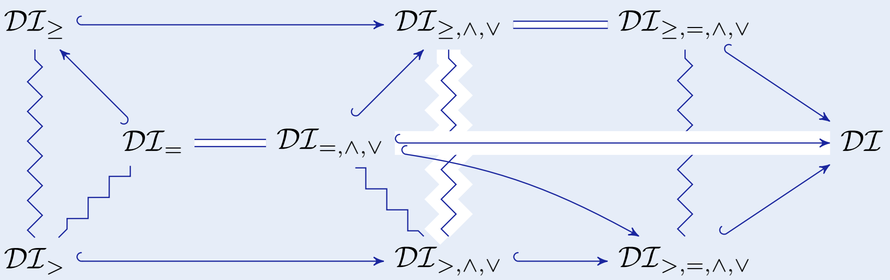


Benefit from norm relations but be mindful of approximation error factors

Outline

Differential Invariance Chart

Theorem (Differential Invariance Chart)



- Rich theory and structure behind differential invariants
- Scrutinize what property can be proved with what invariant
- Use provability sanity checks like open/closed/univariate
- Real differential semialgebraic geometry
- Exploit differential cuts to obtain more knowledge

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The structure of differential invariants and differential cut elimination.
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